

NON-NATURALLY REDUCTIVE EINSTEIN METRICS ON EXCEPTIONAL LIE GROUPS

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ABSTRACT. Given an exceptional compact simple Lie group G we describe new left-invariant Einstein metrics which are not naturally reductive. In particular, we consider fibrations of G over flag manifolds with a certain kind of isotropy representation and we construct the Einstein equation with respect to the induced left-invariant metrics. Then we apply a technique based on Gröbner bases and classify the real solutions of the associated algebraic systems. For the Lie group G_2 we obtain the first known example of a left-invariant Einstein metric, which is not naturally reductive. Moreover, for the Lie groups E_7 and E_8 , we conclude that there exist non-isometric non-naturally reductive Einstein metrics, which are $\text{Ad}(K)$ -invariant by different Lie subgroups K .

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1. INTRODUCTION

In 1979 D'Atri and Ziller [DZ] studied naturally reductive metrics on compact semi-simple Lie groups. They gave a complete classification of such metrics on compact simple Lie groups and described many naturally reductive Einstein metrics. They also asked the following question (cf. [DZ] Remark p. 62):

Question. *Given a compact simple Lie group G , do there exist left-invariant Einstein metrics which are not naturally reductive?*

The first left-invariant Einstein metrics on a compact simple Lie group which are non-naturally reductive, were discovered for SU_n ($n \geq 6$) by K. Mori in 1994 [M]. He considered the Lie group SU_n as a principal bundle over the generalized flag manifold $\text{SU}_n / \text{S}(\text{U}_\ell \times \text{U}_m \times \text{U}_k)$ ($\ell + m + k = n \geq 2$) and then he used the reverse of Kaluza-Klein ansatz to describe new left-invariant Einstein metrics. In 2008, Arvanitoyeorgos, Mori and the second author proved the existence of new non-naturally reductive Einstein metrics for SO_n ($n \geq 11$), Sp_n ($n \geq 3$), E_6 , E_7 and E_8 , using fibrations of a compact simple Lie group over a flag manifold (Kähler C-space) with two isotropy summands (see [AMS]). More recently, Chen and Liang [CL] proved that there is a non-naturally reductive left-invariant Einstein metric on the exceptional Lie group F_4 .

In this paper we describe new non-naturally reductive Einstein metrics on compact simple Lie groups G , which can be viewed as principal bundles over flag manifolds $M = G/K$ with three isotropy summands and second Betti number $b_2(M) = 1$. Hence, the painted Dynkin diagram of M is defined by a pair (Π, Π_K) such that $\Pi \setminus \Pi_K = \{\alpha_{i_o}\}$ with $\text{ht}(\alpha_{i_o}) = 3$ for some simple root α_{i_o} . Here, $\Pi = \{\alpha_1, \dots, \alpha_\ell\}$ is a basis of simple roots and $\text{ht}(\alpha_j)$ is the height (Dynkin mark) of a simple root α_j . Because the heights of a classical compact simple Lie group are bounded by $1 \leq \text{ht}(\alpha_i) \leq 2$ for any $i = 1, \dots, \ell$ (c.f. [GOV]), the examined Lie groups are necessarily *exceptional*, see Table 2. From now on, we shall denote such a Lie group G by $G(\alpha_{i_o}) \equiv G(i_o)$; then one can immediately encode the isotropy subgroup K via the corresponding painted Dynkin diagram. Moreover, the related reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ induces the left-invariant metrics $\langle \cdot, \cdot \rangle$ that we are interested in.

By extending the notation of [AMS], and since in our case the painted black simple root α_{i_o} is never connected with the vertex corresponding to the negative of the maximal root $\tilde{\alpha} := \text{ht}(\alpha_1)\alpha_1 + \dots + \text{ht}(\alpha_\ell)\alpha_\ell$, we agree to say that $G \equiv G(i_o)$ is of Type I_b , II_b , or III_b , if after deleting the black vertex the Dynkin diagram splits into *one*, *two*, or *three* components (subdiagrams), respectively. In [AMS] and for compact simple Lie groups G associated to flag manifolds $M = G/K$ with *two* isotropy summands, it was shown that the *new* non-naturally reductive Einstein metrics appear only for the corresponding classes of Type I_b and II_b (for the Types I_a, II_a, III_a the painted black simple root is connected to $-\tilde{\alpha}$). In particular, for such flag manifolds, there are still Lie groups of Types III_a, III_b (related to $\text{SO}_{2\ell}$, see Theorem 3.3) but these cases have not been examined yet. In this study, we focus on exceptional flag manifolds and provide the existence of *new* left-invariant non-naturally reductive Einstein metrics on simple Lie groups of all 3 types I_b, II_b and III_b .

For convenience, in Table 1 and for any exceptional compact simple Lie group $G \equiv G(i_o)$, we list the number of *non-naturally reductive* left-invariant Einstein metrics found in [AMS], including our new Einstein metrics and the Einstein metric constructed by Chen and Liang [CL] (although it does not fit into our types). We denote this number by $\mathcal{E}(G)_{\text{non-nn}}$ and also state the isotropy subgroup K , the numbers p, q appearing in the reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} = \mathfrak{k}_0 \oplus \cdots \oplus \mathfrak{k}_p \oplus \mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_q$ and the type of $G(i_o)$.

Table 1. Number of non-naturally reductive left-invariant Einstein metrics on G exceptional.

G	notation $G \equiv G(i_o)$	isotropy group K	$p + q$	Type	$\mathcal{E}(G)_{\text{non-nn}}$
G_2	$G_2(2)$	$U_2^t \cong SU_2 \times U_1$	$2 + 3$	I_b	1 new
F_4	$F_4(2)$	$SU_3 \times SU_2 \times U_1$	$3 + 3$	II_b	5 new
	-	SO_8	-	-	1 [CL]
E_6	$E_6(2) \cong E_6(4)$	$SU_5 \times SU_2 \times U_1$	$3 + 2$	II_b	4 [AMS]
	$E_6(3)$	$SU_3 \times SU_2 \times SU_2 \times U_1$	$4 + 3$	III_b	9 new
E_7	$E_7(7)$	$SU_7 \times U_1$	$2 + 2$	I_b	2 [AMS]
	$E_7(2)$	$SO_{10} \times SU_2 \times U_1$	$3 + 2$	II_b	4 [AMS]
	$E_7(5)$	$SU_6 \times SU_2 \times U_1$	$3 + 3$	II_b	5 new
	$E_7(3)$	$SU_5 \times SU_3 \times U_1$	$3 + 3$	II_b	7 new
E_8	$E_8(7)$	$SO_{14} \times U_1$	$2 + 2$	I_b	2 [AMS]
	$E_8(8)$	$SU_8 \times U_1$	$2 + 3$	I_b	3 new
	$E_8(2)$	$E_6 \times SU_2 \times U_1$	$3 + 3$	II_b	5 new

For the description of the Ricci tensor associated to a left-invariant metric on a Lie group $G(i_o)$ of Type I_b , II_b and III_b , we exploit the Lie theoretic description of flag manifolds $M = G(i_o)/K$ and apply a basic method of Riemannian submersions, see for example Lemma 2.2. Notice that the induced left-invariant metrics are given in terms of 5, 6, or 7 parameters, depending on the explicit type of $G = G(i_o)$. Hence, the Ricci tensor is complicated and we often need to consider new fibrations, inducing new left-invariant Riemannian metrics. Then, a comparison of these metrics with the initial one $\langle \cdot, \cdot \rangle$ provides all the necessary information for an explicitly description of the associated Einstein equation, see for example Lemma 4.4 or Lemma 5.3. We mention that this approach leads to a uniform description of the Ricci tensor and it has been successfully applied in a series of works [AMS, AC, CS].

After this step, we proceed to a systematic examination of the algebraic systems defined by the corresponding Einstein equation with respect to $\langle \cdot, \cdot \rangle$. With the aid of computer, we can describe Gröbner bases for algebraic systems and classify real solutions of the corresponding Einstein equation. Finally based on Propositions 4.6, 5.5, 6.4, we deduce which of these solutions induce *new* non-naturally reductive left-invariant Einstein metrics. We summarize our results (up to isometry) in the following theorem (for useful details on the examination of the isometry problem we refer to [CS, AC, C2]).

Theorem 1.1. (a) *The compact simple Lie group G_2 admits at least one non-naturally reductive left-invariant Einstein metric. In particular, this metric is $\text{Ad}(U_2^t)$ -invariant.*

(b) *The compact simple Lie group F_4 admits at least five new non-naturally reductive and non-isometric left-invariant Einstein metrics. These metrics are $\text{Ad}(SU_3 \times SU_2 \times U_1)$ -invariant.*

(c) *The compact simple Lie group E_6 admits at least nine new non-naturally reductive and non-isometric left-invariant Einstein metrics. These metrics are $\text{Ad}(SU_3 \times SU_2 \times SU_2 \times U_1)$ -invariant.*

(d) *The compact simple Lie group E_7 admits at least twelve new non-naturally reductive and non-isometric left-invariant Einstein metrics. Five of these metrics are $\text{Ad}(SU_6 \times SU_2 \times U_1)$ -invariant and the other seven are $\text{Ad}(SU_5 \times SU_3 \times U_1)$ -invariant.*

(e) *The compact simple Lie group E_8 admits at least eight new non-naturally reductive and non-isometric left-invariant Einstein metrics. Three of these metrics are $\text{Ad}(SU_8 \times U_1)$ -invariant and the other five are $\text{Ad}(E_6 \times SU_2 \times U_1)$ -invariant.*

A careful analysis of the Einstein metrics described by Gibbons, Lü and Pope [GLP] on the compact simple Lie group G_2 , shows that they are naturally reductive, which is contrary to the claim appearing in the introduction of [CL]. In particular, the metric described in Theorem 1.1 is, to the best of our knowledge, the first known example of a left-invariant Einstein metric on G_2 , which is *not* naturally reductive. Moreover, another direct conclusion of the present work is the existence of Lie groups G , that is, E_7, E_8 , for which one can construct non-isometric, non-naturally reductive left-invariant Einstein metrics which are $\text{Ad}(K)$ -invariant for different Lie subgroups $K \subset G$. More examples in this direction occur now in combination with the results of [AMS, CL], that is, for the Lie groups F_4, E_6 . We remark, however, that there are also subgroups $K \subset G$ for which the associated reductive decompositions does not induce any new left-invariant Einstein metric,

for $(G = G_2, K = U_2^s)$ and $(G = F_4, K = \mathrm{Sp}_3 \times U_1)$. Eventually, the results stated in Table 1 leads us to conjecture that Lie groups which can be viewed as principal bundles over flag manifolds $M = G/K$ with $b_2(M) = 1$ and $q \geq 4$, admit more non-naturally reductive left-invariant Einstein metrics. By Theorem 3.3, this conjecture may apply for the Lie groups $F_4(3)$, $E_8(3)$, $E_8(6)$, $E_7(4)$, $E_8(4)$, $E_8(5)$ and in the special case $q = 2$, for the classical Lie group $\mathrm{SO}_{2\ell}(\ell - 2)$ with $\ell \geq 5$ (we mean the case that the painted black root is not connected to $-\tilde{\alpha}$).

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2. INVARIANT METRICS AND THE RICCI TENSOR

Let $(M = G/K, g)$ be a compact homogeneous Riemannian manifold, where $G \subset I(M)$ is a closed subgroup of the isometry group and K is the isotropy subgroup at a fixed point $o = eK \in M$. Through this paper we shall denote by $B \equiv B_G$ the negative of the Killing form of $\mathfrak{g} = T_e G$. Without loss of generality we can assume that G acts (almost) effectively on M and moreover that it is connected and semi-simple (see [B]). Fix a B -orthogonal $\mathrm{Ad}(K)$ -invariant complement $\mathfrak{m} \perp \mathfrak{k}$ of \mathfrak{k} in \mathfrak{g} such that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ and $\mathrm{Ad}(K)\mathfrak{m} \subset \mathfrak{m}$. Then, \mathfrak{m} is identified with the tangent space $T_o(G/K)$ ($o = eK \in G/K$), and the isotropy representation $\chi : K \rightarrow \mathrm{SO}(\mathfrak{m})$ of K coincides with the restriction of the adjoint representation $\mathrm{Ad}_G|_K$ on \mathfrak{m} . Thus we may identify g with an $\mathrm{Ad}(K)$ -invariant inner product $(\cdot, \cdot) : \mathfrak{m} \times \mathfrak{m} \rightarrow \mathbb{R}$ on \mathfrak{m} . Traditionally, we call $(M = G/K, g)$ naturally reductive if there exist G and \mathfrak{m} as above, such that the endomorphism $\mathrm{ad}(X) : \mathfrak{m} \rightarrow \mathfrak{m}$ be skew-symmetric with respect to (\cdot, \cdot) for any $X \in \mathfrak{m}$.

In [DZ], D'Atri and Ziller examined naturally reductive metrics among left invariant metrics on compact Lie groups, in particular in the simple case they presented the complete classification of such metrics. Let us recall some details. Consider a compact connected semi-simple Lie group G and let H be a closed subgroup. We shall write $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_p$ for a decomposition of the Lie algebra $\mathfrak{h} = T_e H$ into its centre $\mathfrak{h}_0 := Z(\mathfrak{h})$ and simple ideals \mathfrak{h}_i , for $i = 1, \dots, p$. Let \mathfrak{p} be the orthogonal complement of $\mathfrak{h} \subset \mathfrak{g}$ with respect to B . Then, $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ and $\mathrm{Ad}(H)\mathfrak{p} \subset \mathfrak{p}$. Now, the Lie group $G \times H$ acts almost effectively on G with isotropy group the diagonal group $\Delta(H) = \{(h, h) : h \in H\}$. Thus, G can be viewed as coset $(G \times H)/\Delta(H)$ with $\mathfrak{g} \oplus \mathfrak{h} = \Delta(\mathfrak{h}) \oplus \mathfrak{G}$, where we identify $\mathfrak{G} \cong T_e(G \times H/\Delta H) \cong \mathfrak{g}$ via the linear map $\mathfrak{G} \ni (X, Y) \mapsto X - Y \in \mathfrak{g}$.

Theorem 2.1. ([DZ, Thm. 1, Thm. 3]) *For any inner product b on the centre \mathfrak{h}_0 of \mathfrak{h} , the following left-invariant metric on G is naturally reductive with respect to the action $(g, h)y = gyh^{-1}$ of $G \times H$:*

$$\langle \cdot, \cdot \rangle = u_0 \cdot b|_{\mathfrak{h}_0} + u_1 \cdot B|_{\mathfrak{h}_1} + \cdots + u_p \cdot B|_{\mathfrak{h}_p} + x \cdot B|_{\mathfrak{p}}, \quad (u_0, u_1, \dots, u_p, x \in \mathbb{R}_+).$$

Conversely, if a left invariant metric $\langle \cdot, \cdot \rangle$ on a compact simple Lie group G is naturally reductive, then there exist a closed subgroup $H \subset G$ such that $\langle \cdot, \cdot \rangle$ can be written as above.

The $\mathrm{Ad}(H)$ -invariant orthogonal complement \mathfrak{p} coincides with the tangent space of the homogeneous space G/H , i.e. $\mathfrak{p} \cong T_o G/H$. From now on we assume that $\mathfrak{p} = \mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_q$ defines an orthogonal decomposition of $\mathfrak{p} = T_o(G/K)$ into q irreducible mutually non-equivalent $\mathrm{Ad}(H)$ -modules \mathfrak{p}_j ($j = 1, \dots, q$). We also assume that the ideals \mathfrak{h}_i ($i = 1, \dots, p$) are mutually non-isomorphic with $\dim \mathfrak{h}_0 \leq 1$. With the aim to give a unified expression for the Ricci tensor of a left invariant metric on G and a G -invariant metric on G/H , it is useful to express the decomposition of \mathfrak{g} (and hence also of \mathfrak{h} and \mathfrak{p}) as follows:

$$(2.1) \quad \mathfrak{g} = (\mathfrak{h}_0 \oplus \mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_p) \oplus (\mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_q) = (\mathfrak{m}_0 \oplus \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_p) \oplus (\mathfrak{m}_{p+1} \oplus \cdots \oplus \mathfrak{m}_{p+q})$$

with $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_p \cong \mathfrak{m}_0 \oplus \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_p$ and $\mathfrak{p} = \mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_q \cong \mathfrak{m}_{p+1} \oplus \cdots \oplus \mathfrak{m}_{p+q}$, respectively. Then, the following left-invariant metric on G is in fact $\mathrm{Ad}(H)$ -invariant:

$$(2.2) \quad \begin{aligned} \langle \cdot, \cdot \rangle &= u_0 \cdot B|_{\mathfrak{h}_0} + u_1 \cdot B|_{\mathfrak{h}_1} + \cdots + u_p \cdot B|_{\mathfrak{h}_p} + x_1 \cdot B|_{\mathfrak{p}_1} + \cdots + x_q \cdot B|_{\mathfrak{p}_q}, \\ &= y_0 \cdot B|_{\mathfrak{m}_0} + y_1 \cdot B|_{\mathfrak{m}_1} + \cdots + y_p \cdot B|_{\mathfrak{m}_p} + y_{p+1} \cdot B|_{\mathfrak{m}_{p+1}} + \cdots + y_{p+q} \cdot B|_{\mathfrak{m}_{p+q}} \end{aligned}$$

where $u_a, x_b, y_c \in \mathbb{R}_+$ for any $0 \leq a \leq p$, $1 \leq b \leq q$, $0 \leq y_c \leq p+q$. Similarly, any G -invariant Riemannian metric on G/H is given by

$$(2.3) \quad (\cdot, \cdot) = x_1 \cdot B|_{\mathfrak{p}_1} + \cdots + x_q \cdot B|_{\mathfrak{p}_q} = y_{p+1} \cdot B|_{\mathfrak{m}_{p+1}} + \cdots + y_{p+q} \cdot B|_{\mathfrak{m}_{p+q}}.$$

Let us present now a formula for the Ricci tensor associated to the left-invariant metric $\langle \cdot, \cdot \rangle$ on G defined by (2.2), and describe the same time the Ricci tensor of $M = G/H$ with respect to the G -invariant metric (\cdot, \cdot) given by (2.3) (see also [WZ, PS, AC]). Set from now on $d_i := \dim_{\mathbb{R}} \mathfrak{m}_i$ and let $\{e_{\mu}^i\}_{\mu=1}^{d_i}$ be a B -orthonormal

basis adapted to the decomposition of \mathfrak{g} , i.e., $e_\mu^i \in \mathfrak{m}_i$ for some i , $\mu < \nu$ if $i < j$ with $e_\mu^i \in \mathfrak{m}_i$ and $e_\nu^j \in \mathfrak{m}_j$, for any $0 \leq i, j \leq p+q$. Consider the numbers $A_{\mu\nu}^\xi = B([e_\mu^i, e_\nu^j], e_\xi^k)$ such that $[e_\mu^i, e_\nu^j] = \sum_\xi A_{\mu\nu}^\xi e_\xi^k$, and set

$$A_{ijk} := \begin{bmatrix} k \\ ij \end{bmatrix} = \sum (A_{\mu\nu}^\xi)^2$$

where the sum is taken over all indices μ, ν, ξ with $e_\mu^i \in \mathfrak{m}_i$, $e_\nu^j \in \mathfrak{m}_j$, $e_\xi^k \in \mathfrak{m}_k$. Then, A_{ijk} are independent of the B -orthonormal bases chosen for $\mathfrak{m}_i, \mathfrak{m}_j, \mathfrak{m}_k$, and symmetric in all three indices, i.e. $A_{ijk} = A_{jik} = A_{kij}$.

Note that $\text{Ric}_{\langle \cdot, \cdot \rangle}$ is also an $\text{Ad}(H)$ -invariant symmetric (covariant) tensor. In particular, since $\mathfrak{m} \not\cong \mathfrak{m}_j$ for any $0 \leq i \neq j \leq p+q$, it is $\text{Ric}_{\langle \cdot, \cdot \rangle}(\mathfrak{m}_i, \mathfrak{m}_j) = 0$ for $i \neq j$, that is, the Ricci tensor is diagonal. Now, for any $k = 0, \dots, p+q$, the set $\{e_\mu^k / \sqrt{y_k}\}_{\mu=1}^{d_k}$ forms an $\langle \cdot, \cdot \rangle$ -orthonormal basis of \mathfrak{m}_k . Consider real numbers r_k defined by $r_k := \text{Ric}_{\langle \cdot, \cdot \rangle}(e_\mu^k / \sqrt{y_k}, e_\mu^k / \sqrt{y_k}) = (1/y_k) \text{Ric}_{\langle \cdot, \cdot \rangle}(e_\mu^k, e_\mu^k)$. Then, we have $\text{Ric}_{\langle \cdot, \cdot \rangle} = \sum_{k=0}^{p+q} y_k r_k B|_{\mathfrak{m}_k}$.

Lemma 2.2. ([PS, AMS]) *Let G be a compact connected semi-simple Lie group endowed with the left-invariant metric $\langle \cdot, \cdot \rangle$ given by (2.2). Then the components r_0, r_1, \dots, r_{p+q} of the Ricci tensor $\text{Ric}_{\langle \cdot, \cdot \rangle}$ associated to $\langle \cdot, \cdot \rangle$, are expressed as follows:*

$$r_k = \frac{1}{2y_k} + \frac{1}{4d_k} \sum_{j,i} \frac{y_k}{y_j y_i} \begin{bmatrix} k \\ ji \end{bmatrix} - \frac{1}{2d_k} \sum_{j,i} \frac{y_j}{y_k y_i} \begin{bmatrix} j \\ ki \end{bmatrix} \quad (k = 0, 1, \dots, p+q).$$

Here, the sums are taken over all $i, j = 0, 1, \dots, p+q$. In particular, for each k it holds that

$$\sum_{i,j} \begin{bmatrix} j \\ ki \end{bmatrix} = \sum_{i,j} A_{kij} = d_k := \dim_{\mathbb{R}} \mathfrak{m}_k.$$

For $k = p+1, \dots, p+q$ and by considering the sums appearing in the expression of r_k only for i, j with $p+1 \leq i, j \leq p+q$, one obtains the components $\bar{r}_{p+1}, \dots, \bar{r}_{p+q}$ of the Ricci tensor \bar{r} of the G -invariant metric (\cdot, \cdot) on G/H defined by (2.3).

3. FLAG MANIFOLDS WITH SECOND BETTI NUMBER $b_2(M) = 1$ AND LIE GROUPS OF TYPE I, II, OR III

Let G be a compact connected simple Lie group with finite centre and Dynkin diagram $\Gamma(\Pi)$, where Π denotes a basis of simple roots. We are interested in K -principal fibrations of G over flag manifolds $M = G/K$ with the aim to build left-invariant metrics on G via a metric on the base G/K and a metric on the fiber K . For a Lie theoretic description of flag manifolds in terms of painted Dynkin diagrams we refer to [AP, AC, C2, CS].

3.1. Lie groups of Type I, II or III. It is well-known (see [C2, C1, CS]) that by painting black on $\Gamma(\Pi)$ the vertex of a simple root, say α_{i_o} for some $1 \leq i_o \leq \ell$, we define a flag manifold $M = G/K$ whose isotropy representation decomposes into $q := \text{ht}(\alpha_{i_o}) \in \mathbb{Z}_+$ mutually inequivalent, irreducible $\text{Ad}(K)$ -submodules, i.e.

$$\mathfrak{p} = T_o M = \mathfrak{p}_1 \oplus \dots \oplus \mathfrak{p}_q.$$

In fact, these are the flag manifolds $M = G/K$ with $b_2(M) = 1$ and their classification can be found for example in [CS, Table 1]. Because $M = G/K$ is just defined by fixing a simple root α_{i_o} , we will denote the corresponding Lie group G by $G(i_o) \equiv G(\alpha_{i_o})$. We shall write $G(i) \cong G(j)$ when the flag manifolds obtained by the subsets $\Pi_M = \{\alpha_i\}$ and $\Pi'_M = \{\alpha_j\}$, are isomorphic.

Given the Dynkin diagram $\Gamma(\Pi)$ of G , by deleting a vertex we obtain at most three components (sub-diagrams), with the 3-component case appearing only for the $G \cong E_6, E_7, E_8$ or $\text{SO}_{2\ell}$. These components correspond to the Dynkin diagrams which define the semi-simple part of K . Hence, we have the same number of simple ideals in \mathfrak{k} and components in $\Gamma(\Pi)$ after deleting α_{i_o} , and as long as this number increases it brakes up the symmetry of $G(i_o)$. The centre $\mathfrak{k}_0 \cong \mathfrak{u}_1$ corresponds to the black vertex and we shall use also a double circle \odot to denote the negative of the maximal root $\tilde{\alpha} = \sum_{i=1}^{\ell} \text{ht}(\alpha_i) \alpha_i$, with respect to the fixed basis Π . Finally, in the splitting $\mathfrak{k} = \mathfrak{k}_0 \oplus \mathfrak{k}_1 \oplus \dots \oplus \mathfrak{k}_p$ of the isotropy subalgebra \mathfrak{k} into its centre and simple ideals, we agree to denote by \mathfrak{k}_1 the subalgebra whose Dynkin diagram is connected with $-\tilde{\alpha}$.

Definition 3.1. We separate the compact simple Lie groups $G \equiv G(i_o) \equiv G(\alpha_{i_o})$ ($1 \leq i_o \leq \ell$) into three types (and similarly the corresponding flag manifolds $M = G/K$ with $b_2(M) = 1$), namely Type I, II, or III, depending on the number of components (namely one, two or three components) on the Dynkin diagram $\Gamma(\Pi)$ of $G(i_o)$, after deleting the black vertex corresponding to α_{i_o} . In particular, we shall write $I(q)$, $II(q)$, or $III(q)$, where $q = \text{ht}(\alpha_{i_o})$ coincides with the number of the isotropy summands of $M = G/K$. We further divide each of these three types, into two subclasses by inserting a subscript a or b , e.g. Type $I_a(q)$ or Type $I_b(q)$, depending whether the black vertex is connected to $-\tilde{\alpha}$, or not.

Let us classify now the compact simple Lie groups with respect to the above 6 types. We mention that maybe not all the types appear for any q with $1 \leq q \leq 6$, or for some of them just a few Lie groups exist. For example

Proposition 3.2. *The unique compact simple Lie group $G = G(i_o) = G(\alpha_{i_o})$ of Type $III_a(q)$ for some q ($1 \leq q \leq 6$), is the Lie group $SO_8(2)$ with $q = 2$.*

Proof. By definition, a Lie group $G(i_o)$ of Type $III(q)$ ($1 \leq q \leq 6$) can only be isometric to one of $SO_{2\ell}$, E_6 , E_7 , E_8 , namely: $SO_{2\ell}(\ell-2)$, $E_6(3)$, $E_7(4)$ and $E_8(5)$. In fact, due to the form of extended Dynkin diagrams (see [AMS, GOV]), these groups are always of Type $III_b(q)$, i.e. in any case the painted black simple root is not connected with $-\tilde{\alpha}$, except $SO_8(2)$ which is of Type $III_a(q)$ with $q = 2$; the corresponding flag manifold is $SO_8/(SU_2 \times SU_2 \times SU_2 \times U_1)$. For the general case $SO_{2\ell}(\ell-2)$ with $\ell \geq 5$, the associated flag manifold $SO_{2\ell}/(SU_{\ell-2} \times SU_2 \times SU_2 \times U_1)$ is of Type $III_b(q)$ with $q = 2$. For $\ell = 4$ it is still $q = 2$ but as we explained before, in this case the type changes. For the exceptional Lie groups of type $III(q)$, it is $q = 3, 4$ and 6 , respectively, with $E_6/(SU_3 \times SU_3 \times SU_2 \times U_1)$, $E_7/(SU_4 \times SU_3 \times SU_2 \times U_1)$, and $E_8/(SU_5 \times SU_3 \times SU_2 \times U_1)$ being the corresponding Kähler C-spaces, see also [CS, Table 1]. Their exact type is III_b . The Lie group $E_6(3)$ will be examined in Section 6. \square

Theorem 3.3. *Let $M = G/K$ be a generalized flag manifold with $b_2(M) = 1$, where G is a compact, connected, simple Lie group. The classification of $M = G/K$ (and hence of G) with respect to the Types $I_a(q)$, $I_b(q)$, $II_a(q)$, $II_b(q)$, $III_a(q)$, and $III_b(q)$, for $q \geq 1$, is given as follows:*

Type	compact simple Lie group $G \equiv G(i_o) \equiv G(\alpha_{i_o})$
Type $I_b(1)$	$SO_{2\ell+1}(1)$, $Sp_\ell(\ell)$, $SO_{2\ell}(1)$, $SO_{2\ell}(\ell-1) \cong SO_{2\ell}(\ell)$, $E_6(1) \cong E_6(5)$, $E_7(1)$
Type $II_b(1)$	$SU_\ell(\rho)$ ($1 \leq \rho \leq \ell-1$)
Type $I_a(2)$	$Sp_\ell(1)$, $E_6(6)$, $E_7(6)$, $E_8(1)$, $F_4(1)$, $G_2(1)$
Type $I_b(2)$	$E_7(7)$, $E_8(7)$, $F_4(4)$
Type $II_a(2)$	$SO_{2\ell+1}(2)$, $SO_{2\ell}(2)$
Type $II_b(2)$	$SO_{2\ell+1}(\rho)$ ($2 \leq \rho \leq \ell-1$), $Sp_\ell(\rho)$ ($2 \leq \rho \leq \ell-1$), $SO_{2\ell}(\rho)$ ($3 \leq \rho \leq \ell-3$), $E_6(2) \cong E_6(4)$, $E_7(2)$
Type $III_a(2)$	$SO_8(2)$
Type $III_b(2)$	$SO_{2\ell}(\ell-2)$ ($\ell \geq 5$)
Type $I_b(3)$	$G_2(2)$, $E_8(8)$
Type $II_b(3)$	$F_4(2)$, $E_7(3)$, $E_7(5)$, $E_8(2)$
Type $III_b(3)$	$E_6(3)$
Type $II_b(4)$	$F_4(3)$, $E_8(3)$, $E_8(6)$
Type $III_b(4)$	$E_7(4)$
Type $II_b(5)$	$E_8(4)$
Type $III_b(6)$	$E_8(5)$

Proof. The case $q = 2$ has been already examined in [AMS], except the Types $III_a(2)$ and $III_b(2)$ which we include here. These result exhaust all possible types with $q = 2$. Now, due to the form of the maximal root $\tilde{\alpha}$ we need to examine the cases $q = 1, 3, 4, 5, 6$. For $q = 1$, $M = G/K$ is isometric to a compact isotropy irreducible Hermitian symmetric space. The classification of flag manifolds with three isotropy summands and $b_2(M) = 1$ was given in [K] and this with four isotropy summands was obtained in [AC, C1]. The cases $q = 5, 6$ appear only for E_8 , see [CS]. Now, the presented results are a combination of the Definition 3.1 and the (extended) painted Dynkin diagrams associated to flag manifolds with $b_2(M) = 1$. \square

3.2. Flag manifolds with three isotropy summands and $b_2(M) = 1$. From now on we focus on flag manifolds $M = G/K$ with three isotropy summands $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3$ and second Betti number $b_2(M) = 1$. Hence, $M = G/K$ is defined by a pair (Π, Π_K) such that $\Pi_M := \Pi \setminus \Pi_K = \{\alpha_{i_o}\}$ with $\text{ht}(\alpha_{i_o}) = 3$, see also [K, AnC]. As one can read from Theorem 3.3, such pairs (Π, Π_K) exist only for an exceptional simple Lie group, in particular $G(i_o) \equiv G(\alpha_{i_o})$ must be isometric to one of the following Lie groups: $G_2(2)$, $E_8(8)$, $F_4(2)$, $E_7(3)$, $E_7(5)$, $E_8(2)$, and $E_6(3)$. For these groups we present in Table 2 the associated flag manifolds $G(i_o)/K$ (via their painted Dynkin diagrams) and we state the necessary dimensions $D_i := \dim_{\mathbb{R}} \mathfrak{p}_i$ for $i = 1, 2, 3$. By U_2^l (resp. U_2^s) we denote the Lie group isomorphic to $U_2 \cong SU_2 \times U_1$, generated by the long (resp. short) root of the basis $\Pi = \{\alpha_1 = e_2 - e_3, \alpha_2 = -e_2\}$ associated to the root system of G_2 . For this case, recall that $\|\alpha_1\| = \sqrt{3}\|\alpha_2\|$ and $\tilde{\alpha} = 2\alpha_1 + 3\alpha_2$. The highest roots of the exceptional groups F_4, E_6, E_7 and E_8 with respect to the used fundamental bases (see [C1, GOV]) are given as follows (we write only the heights ($\text{ht}(\alpha_1), \dots, \text{ht}(\alpha_\ell)$)):

$$F_4 : (2, 3, 4, 2), \quad E_6 : (1, 2, 3, 2, 1, 2), \quad E_7 : (1, 2, 3, 4, 3, 2, 2), \quad E_8 : (2, 3, 4, 5, 6, 2, 4, 3).$$

Remark 3.4. An explicit computation of the triples A_{ijk} associated to the reductive decomposition of the Lie algebra \mathfrak{g} of $G = G(i_o)$, is possible via their definition. This method access a description in terms of the structure constants of the corresponding Lie algebra (see also [S]). In general, we avoid this technique, since we need to repeat it separately for any Lie group of each type and it becomes combinatorial. The same time, Theorem 3.3 and the classification of the simple Lie groups into different types $I(q)$, $II(q)$, $III(q)$, ($1 \leq q \leq 6$) allows us to compute A_{ijk} explicitly, in a generalized way. This means that often Lie groups of exactly the same type can be treated simultaneously; in this situation one can describe the Ricci tensor and construct the Einstein equation for the fixed left-invariant metric only once, and not for any Lie group separately; we refer to [AMS] for the Types $I(q)$, $II(q)$ and $q = 2$ and see Sections 4, 5 and 6 for the Types $I(q)$, $II(q)$ and $III(q)$ with $q = 3$, respectively.

Table 2. Painted Dynkin diagrams of flag manifolds $M = G/K$ with $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3$ and $b_2(M) = 1$.

	G	(Π, Π_K)	K	(D_1, D_2, D_3)
Type $I_b(3)$				
	$G_2(2)$		U_2^l	$(4, 2, 4)$
	$E_8(8)$		$SU_8 \times U_1$	$(112, 56, 16)$
Type $II_b(3)$				
	$F_4(2)$		$SU_3 \times SU_2 \times U_1$	$(22, 12, 4)$
	$E_7(3)$		$SU_5 \times SU_3 \times U_1$	$(60, 30, 10)$
	$E_7(5)$		$SU_6 \times SU_2 \times U_1$	$(60, 30, 4)$
	$E_8(2)$		$E_6 \times SU_2 \times U_1$	$(108, 54, 4)$
Type $III_b(3)$				
	$E_6(3)$		$SU_3 \times SU_3 \times SU_2 \times U_1$	$(36, 18, 4)$

4. LEFT-INVARIANT NON-NATURALLY REDUCTIVE EINSTEIN METRICS ON LIE GROUPS OF TYPE $I_b(3)$

Let $G \cong G(i_o)$ be a compact connected Lie groups of Type $I_b(3)$. Then G is isometric to $G_2(2)$ or $E_8(8)$. From now on we shall write $\mathfrak{g} = T_e G$ for the corresponding Lie algebra.

4.1. The Ricci tensor. For a Lie group $G \cong G(i_o)$ of Type $I_b(3)$ consider the orthogonal decomposition

$$(4.1) \quad \mathfrak{g} = \mathfrak{k}_0 \oplus \mathfrak{k}_1 \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3 = \mathfrak{m}_0 \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3 \oplus \mathfrak{m}_4.$$

This is a reductive decomposition of \mathfrak{g} of the form (2.1) and due to (2.2), a left-invariant metric on $G \cong G(i_o)$ is given by

$$(4.2) \quad \langle \cdot, \cdot \rangle = u_0 \cdot B|_{\mathfrak{k}_0} + u_1 \cdot B|_{\mathfrak{k}_1} + x_1 \cdot B|_{\mathfrak{p}_1} + x_2 \cdot B|_{\mathfrak{p}_2} + x_3 \cdot B|_{\mathfrak{p}_3} = y_0 \cdot B|_{\mathfrak{m}_0} + y_1 \cdot B|_{\mathfrak{m}_1} + y_2 \cdot B|_{\mathfrak{m}_2} + y_3 \cdot B|_{\mathfrak{m}_3} + y_4 \cdot B|_{\mathfrak{m}_4}$$

for some positive numbers $u_0, u_1, x_i, y_j \in \mathbb{R}_+$. This metric is also $\text{Ad}(K)$ -invariant and since $\mathfrak{m}_i \not\cong \mathfrak{m}_j$ for all $2 \leq i \neq j \leq 4$, any G -invariant metric on the base space $M = G/K$ is of the form

$$(\cdot, \cdot) = x_1 \cdot B|_{\mathfrak{p}_1} + x_2 \cdot B|_{\mathfrak{p}_2} + x_3 \cdot B|_{\mathfrak{p}_3} = y_2 \cdot B|_{\mathfrak{m}_2} + y_3 \cdot B|_{\mathfrak{m}_3} + y_4 \cdot B|_{\mathfrak{m}_4}.$$

Remark 4.1. For $G_2(2)$ it is $\mathfrak{k}_0 \cong \mathfrak{u}_1$, $\mathfrak{k}_1 \cong \mathfrak{su}_2$ and $d_0 = 1$, $d_1 = 3$, $d_2 = 4$, $d_3 = 2$, $d_4 = 4$. For $E_8(8)$ it is $\mathfrak{k}_0 \cong \mathfrak{u}_1$, $\mathfrak{k}_1 \cong \mathfrak{su}_8$ and $d_0 = 1$, $d_1 = 63$, $d_2 = 112$, $d_3 = 56$, $d_4 = 16$.

Proposition 4.2. *For the reductive decomposition (4.1) associated to the compact simple Lie group $G_2(2)$ and for the left-invariant metric given by (4.2), the non-zero structure constants A_{ijk} ($0 \leq i, j, k \leq 4$) are the following (and their symmetries): $A_{022}, A_{033}, A_{044}, A_{111}, A_{122}, A_{144}, A_{223}, A_{234}$. This hold also for $E_8(8)$, but in this case one has in addition $A_{133} \neq 0$.*

Proof. The proof is based on Lie theoretic arguments and it is similar with the one which we shall present for Proposition 5.1. Since the latter case is a bit more complicated, we state here only a few details and we refer to this proof for an extensive description of the different techniques that can be applied. First notice that

$$[\mathfrak{m}_2, \mathfrak{m}_2] \subset \mathfrak{m}_3 \oplus \mathfrak{k}, \quad [\mathfrak{m}_2, \mathfrak{m}_3] \subset \mathfrak{m}_2 \oplus \mathfrak{m}_4, \quad [\mathfrak{m}_2, \mathfrak{m}_4] \subset \mathfrak{m}_3, \quad [\mathfrak{m}_3, \mathfrak{m}_3] \subset \mathfrak{k}, \quad [\mathfrak{m}_3, \mathfrak{m}_4] \subset \mathfrak{m}_2, \quad [\mathfrak{m}_4, \mathfrak{m}_4] \subset \mathfrak{k}.$$

These inclusions occur since the base space of the K -principal bundle $G \rightarrow M = G/K$ is a flag manifold with $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3 (= \mathfrak{m}_2 \oplus \mathfrak{m}_3 \oplus \mathfrak{m}_4)$ and $b_2(M) = 1$ (see [AnC, p. 1593] and [CS, p. 674] for the general case). Notice also that $[\mathfrak{m}_0, \mathfrak{m}_j] \subset \mathfrak{m}_j$ for $j = 2, 3, 4$. Moreover, for $G_2(2)$ one can show that

$$[\mathfrak{m}_1, \mathfrak{m}_1] \subset \mathfrak{m}_1, \quad [\mathfrak{m}_1, \mathfrak{m}_2] \subset \mathfrak{m}_2, \quad [\mathfrak{m}_1, \mathfrak{m}_4] \subset \mathfrak{m}_4,$$

but $[\mathfrak{m}_1, \mathfrak{m}_3] = 0$. Hence $A_{113} = 0$. Indeed, the highest weight of $\mathfrak{k}_1 = \mathfrak{su}_2 = \mathfrak{m}_1$ is α_1 and the highest weight of \mathfrak{m}_3 is $\alpha_1 + 2\alpha_2$. By using the corresponding root vectors we see that $[E_{\pm\alpha_1}, E_{\pm(\alpha_1+2\alpha_2)}] = 0$, because $\pm\alpha_1 \pm (\alpha_1 + 2\alpha_2)$ is not a root of G_2 . The highest weight of \mathfrak{m}_4 is the maximal root $\tilde{\alpha}$. Then $E_{\tilde{\alpha}-\alpha_1} = E_{\alpha_1+3\alpha_2} \in \mathfrak{m}_4$ and we finally conclude that $[\mathfrak{m}_1, \mathfrak{m}_4] \subset \mathfrak{m}_4$. For $E_8(8)$ one gets that $[\mathfrak{m}_1, \mathfrak{m}_j] \subset \mathfrak{m}_j$, for any $j = 2, 3, 4$, so A_{133} is non-zero in this case. Another approach, appropriate for the examination of A_{133} , is based on the orthogonality of roots, see also Proposition 5.1. \square

By applying now Lemma 2.2 we get that

Corollary 4.3. *On $(E_8(8), \langle \cdot, \cdot \rangle)$, the components r_i of the Ricci tensor $\text{Ric}_{\langle \cdot, \cdot \rangle}$ associated to the left-invariant metric $\langle \cdot, \cdot \rangle$ given by (4.2), are described as follows*

$$\left\{ \begin{array}{l} r_0 = \frac{u_0}{4d_0} \left(\frac{A_{022}}{x_1^2} + \frac{A_{033}}{x_2^2} + \frac{A_{044}}{x_3^2} \right), \quad r_1 = \frac{A_{111}}{4d_1} \cdot \frac{1}{u_1} + \frac{u_1}{4d_1} \left(\frac{A_{122}}{x_1^2} + \frac{A_{133}}{x_2^2} + \frac{A_{144}}{x_3^2} \right), \\ r_2 = \frac{1}{2x_1} - \frac{1}{2d_2} \cdot \frac{1}{x_1^2} \left(u_0 \cdot A_{022} + u_1 \cdot A_{122} + x_2 \cdot A_{223} \right) + \frac{A_{234}}{2d_2} \left(\frac{x_1}{x_2x_3} - \frac{x_2}{x_1x_3} - \frac{x_3}{x_1x_2} \right), \\ r_3 = \frac{1}{2x_2} - \frac{1}{2d_3x_2^2} \left(u_0 \cdot A_{033} + u_1 \cdot A_{133} \right) + \frac{A_{223}}{4d_3} \left(\frac{x_2}{x_1^2} - \frac{2}{x_2} \right) + \frac{A_{234}}{2d_3} \left(\frac{x_2}{x_1x_3} - \frac{x_1}{x_2x_3} - \frac{x_3}{x_1x_2} \right), \\ r_4 = \frac{1}{2x_3} - \frac{1}{2d_4} \cdot \frac{1}{x_3^2} \left(u_0 \cdot A_{044} + u_1 \cdot A_{144} \right) + \frac{A_{234}}{2d_4} \left(\frac{x_3}{x_1x_2} - \frac{x_1}{x_2x_3} - \frac{x_2}{x_1x_3} \right). \end{array} \right.$$

The corresponding Ricci components of $G_2(2)$ occur by the same expressions, by setting however $A_{133} = 0$.

4.2. The structure constants. We proceed with the computation of the non-zero structure constants. Two of them, namely A_{234} and A_{223} can be directly obtained using the unique Kähler-Einstein metric that any flag manifold $M = G/K$ with $b_2(M) = 1$ and $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3$ admits. By inserting the values $x_1 = 1$, $x_2 = 2$, $x_3 = 3$ in the homogeneous Einstein equation $\{\bar{r}_2 - \bar{r}_3 = 0, \bar{r}_3 - \bar{r}_4 = 0\}$ where \bar{r}_i ($i = 2, 3, 4$) are the components of Ricci tensor $\text{Ric}_{\langle \cdot, \cdot \rangle}$ associated to the Kähler C-space $(M = G/K, \langle \cdot, \cdot \rangle)$ and solving this system with respect to A_{223}, A_{234} , one computes that (see [AnC])

$$(4.3) \quad A_{223} = \frac{d_2d_3 + 2d_2d_4 - d_3d_4}{d_2 + 4d_3 + 9d_4}, \quad A_{234} = \frac{(d_2 + d_3)d_4}{d_2 + 4d_3 + 9d_4}.$$

For $G_2(2)$ this gives $A_{223} = 2/3$ and $A_{234} = 1/2$, and for $E_8(8)$ we get $A_{223} = 56/3$, $A_{234} = 28/5$. Now, by Lemma 2.2 we also compute that

$$(4.4) \quad G_2(2) : \left\{ \begin{array}{l} d_0 = 1 = A_{022} + A_{033} + A_{044}, \\ d_1 = 3 = A_{111} + A_{122} + A_{144}, \\ d_2 = 4 = 2(A_{022} + A_{122} + A_{223} + A_{234}), \\ d_3 = 2 = 2A_{033} + A_{223} + 2A_{234}, \\ d_4 = 4 = 2(A_{044} + A_{144} + A_{234}). \end{array} \right. \quad E_8(8) : \left\{ \begin{array}{l} 1 = A_{022} + A_{033} + A_{044}, \\ 63 = A_{111} + A_{122} + A_{133} + A_{144}, \\ 112 = 2(A_{022} + A_{122} + A_{223} + A_{234}), \\ 56 = 2A_{033} + 2A_{133} + A_{223} + 2A_{234}, \\ 16 = 2(A_{044} + A_{144} + A_{234}). \end{array} \right.$$

Lemma 4.4. *For the reductive decomposition (4.1) and for the left-invariant metric $\langle \cdot, \cdot \rangle$ on $G_2(2)$ given by (4.2), the non-zero A_{ijk} are given explicitly as follows:*

$$A_{022} = 1/12, \quad A_{033} = 1/6, \quad A_{044} = A_{122} = A_{144} = 3/4, \quad A_{111} = 3/2, \quad A_{223} = 2/3, \quad A_{234} = 1/2.$$

For $E_8(8)$, the corresponding non-zero triples A_{ijk} attain the values

$$\begin{aligned} A_{022} &= 7/30, & A_{044} &= 3/10, & A_{111} &= 84/5, & A_{122} &= 63/2, & A_{144} &= 21/10, \\ A_{033} &= 7/15, & A_{133} &= 63/5, & A_{223} &= 56/3, & A_{234} &= 28/5. \end{aligned}$$

Proof. Case of $G_2(2)$. The fourth equation in (4.4) implies that $A_{033} = (1/2)(d_3 - A_{223} - 2A_{234}) = 1/6$. Therefore, one gets a system with four equations and five unknowns, namely $A_{022}, A_{044}, A_{111}, A_{122}, A_{144}$. Consider the Killing metric defined by $y_i = 1$ for any $i = 0, \dots, 4$. This is a bi-invariant Einstein metric on G_2 (hence also left-invariant), and thus it satisfies the system of equations $\{r_0 - r_1 = 0, r_1 - r_2 = 0, r_2 - r_3 = 0, r_3 - r_4 = 0\}$. Solutions of this system are given by $A_{111} = 3/2$ and

$$(4.5) \quad A_{044} = 5/6 - A_{022}, \quad A_{122} = 1/96(80 - 96A_{022}), \quad A_{144} = 2/3 + A_{022}.$$

Notice that the first equation coincides with the first equation in (4.4), after replacing the previous value of A_{033} , the second equation with the third equation in (4.4) after inserting the values of A_{223}, A_{234} and finally the last one is the same with the second or the fifth equation in (4.4), after substituting the values of A_{122} and A_{044} , respectively, given above. Hence, it suffices to compute some of the triples appearing in (4.5). Set

$$\mathfrak{g} = \mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \mathfrak{n}, \quad \mathfrak{h}_1 := \mathfrak{k}_0 \oplus \mathfrak{p}_2, \quad \mathfrak{h}_2 := \mathfrak{k}_1, \quad \mathfrak{n} := \mathfrak{p}_1 \oplus \mathfrak{p}_3.$$

The space $\mathfrak{h}_1 = \mathfrak{k}_0 \oplus \mathfrak{p}_2 \cong \mathfrak{m}_0 \oplus \mathfrak{m}_3$ is a Lie subalgebra of $\mathfrak{g} = \mathfrak{g}_2$ isomorphic to \mathfrak{su}_2 and the same is true for $\mathfrak{h}_2 = \mathfrak{k}_1 \cong \mathfrak{m}_1$. Thus $\mathfrak{h} := \mathfrak{h}_1 \oplus \mathfrak{h}_2$ is a Lie subalgebra of \mathfrak{g} isomorphic to $\mathfrak{su}_2 \oplus \mathfrak{su}_2$, and the above decomposition is $\text{Ad}(H)$ -invariant. Here, $H = \text{SO}_4$ is the connected Lie group corresponding to \mathfrak{h} . In this way we define a fibration

$$\mathbb{C}P^1 = \text{SO}_4 / U_2 \rightarrow G/K = G_2 / U_2 \rightarrow G/H = G_2 / \text{SO}_4,$$

with the base space being irreducible symmetric space. Left-invariant metrics on G_2 are given now by $\langle\langle \cdot, \cdot \rangle\rangle = w_1 \cdot B|_{\mathfrak{h}_1} + w_2 \cdot B|_{\mathfrak{h}_2} + w_3 \cdot B|_{\mathfrak{n}}$, with $w_1, w_2, w_3 \in \mathbb{R}_+$. This metric is $\text{Ad}(H)$ -invariant, thus also $\text{Ad}(K)$ -invariant and for $w_1 = u_0 = x_2$, $w_2 = u_1$ and $w_3 = x_1 = x_3$ it coincides with the left-invariant metric $\langle \cdot, \cdot \rangle$ described before. For these values, the same holds for the corresponding Ricci tensors $\text{Ric}_{\langle \cdot, \cdot \rangle}$ and $\text{Ric}_{\langle\langle \cdot, \cdot \rangle\rangle}$. In particular, let us denote by $\tilde{\mathbf{r}}_1, \tilde{\mathbf{r}}_2, \tilde{\mathbf{r}}_3$ the components of $\text{Ric}_{\langle\langle \cdot, \cdot \rangle\rangle}$ with respect to the new left-invariant metric $\langle\langle \cdot, \cdot \rangle\rangle$. Then, for $u_0 = w_1$, $x_2 = w_1$, $u_1 = w_2$, $x_1 = w_3$ and $x_3 = w_3$, it holds that

$$\tilde{\mathbf{r}}_1 = r_0 = r_3, \quad \tilde{\mathbf{r}}_2 = r_1, \quad \tilde{\mathbf{r}}_3 = r_2 = r_4.$$

By using the relation $r_2 = r_4$ we see that $A_{022} + A_{223} = A_{044}$ and thus we get $A_{022} = 1/12$. Using relations (4.5) we easily conclude.

Case of $E_8(8)$. By Proposition 4.2 for $E_8(8)$ it is $A_{133} \neq 0$. Therefore, by (4.4) and (4.3) we cannot immediately compute A_{033} (as we did for $G_2(2)$). We need to construct more equations. Set

$$\mathfrak{e}_8 = \mathfrak{g} = \mathfrak{u} \oplus \mathfrak{r}, \quad \mathfrak{u} := \mathfrak{k}_0 \oplus \mathfrak{k}_1 \oplus \mathfrak{p}_3 = \mathfrak{k} \oplus \mathfrak{p}_3, \quad \mathfrak{r} := \mathfrak{p}_1 \oplus \mathfrak{p}_2$$

Then, the following inclusions hold $[\mathfrak{u}, \mathfrak{u}] \subset \mathfrak{u}$, $[\mathfrak{u}, \mathfrak{r}] \subset \mathfrak{r}$, i.e. $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{r}$ is a reductive decomposition of the homogeneous space G/U , where U is the connected Lie subgroup generated by the Lie algebra \mathfrak{u} . Since $\mathfrak{u} \subset \mathfrak{g}$ we get the fibration $U/K \rightarrow G/K \rightarrow G/U$ where $G/K = E_8 / (U_1 \times \text{SU}_8)$ and $\mathfrak{u} \cong \mathfrak{su}(9)$. In full details

$$\mathbb{C}P^8 = \text{SU}_9 / U_8 \rightarrow E_8 / (U_1 \times \text{SU}_8) \rightarrow E_8 / \text{SU}_9$$

where the base space is (strongly) isotropy irreducible but not a symmetric space. Consider now left-invariant metrics on E_8 given by $\langle\langle \cdot, \cdot \rangle\rangle = z_1 \cdot B|_{\mathfrak{u}} + z_2 \cdot B|_{\mathfrak{r}}$ with $z_1, z_2 \in \mathbb{R}_+$. This is an $\text{Ad}(U)$ -invariant metric and for $z_1 = y_0 = y_1 = x_3$, $z_2 = x_1 = x_2$ coincides with the left-invariant metric $\langle \cdot, \cdot \rangle$, defined by (4.2). For these values the associated Ricci components of $(E_8(8), \langle \cdot, \cdot \rangle)$ are such that $r_0 = r_1 = r_4$ and $r_2 = r_3$. Hence, we get for example $63A_{022} + 63A_{033} - A_{122} - A_{133} = -441 + 10A_{122} + 10A_{133}$, $63A_{044} - A_{111} - A_{144} = -1512 + 315A_{044} + 40A_{111} + 355A_{144}$ and $28 - 5A_{022} + 10A_{033} - 5A_{122} + 10A_{133} = 0$. After solving this system of equations, together with the equations obtained by the Killing metric, we get that

$$(4.6) \quad \begin{aligned} &A_{044} = 3/10, & A_{144} &= 21/10, & A_{122} &= 476/15 - A_{022}, \\ &A_{111} &= 84/5, & A_{033} &= 7/10 - A_{022}, & A_{133} &= 371/30 + A_{022}. \end{aligned}$$

Now it is sufficient to compute A_{022} . Set

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}, \quad \mathfrak{h} := \mathfrak{k}_0 \oplus \mathfrak{k}_1 \oplus \mathfrak{p}_2, \quad \mathfrak{n} := \mathfrak{p}_1 \oplus \mathfrak{p}_3.$$

In this case one can easily prove that $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$, $[\mathfrak{h}, \mathfrak{n}] \subset \mathfrak{n}$ and $[\mathfrak{n}, \mathfrak{n}] \subset \mathfrak{h}$. For dimensional reasons it is $\mathfrak{h} \cong \mathfrak{so}(16)$, or in other words, we get a fibration $H/K \rightarrow G/K \rightarrow G/H$ with $\mathfrak{n} \cong T_{o'}G/H$ ($o' = eH \in G/H$). Both the fiber H/K and the base space G/K are irreducible symmetric spaces, in particular the fibration

$$\mathrm{SO}_{16}/\mathrm{U}_8 \rightarrow \mathrm{E}_8/(\mathrm{U}_1 \times \mathrm{SU}_8) \rightarrow \mathrm{E}_8/\mathrm{SO}_{16}.$$

is the twistor fibration of the flag manifold $\mathrm{E}_8/\mathrm{U}_8 \cong \mathrm{E}_8/(\mathrm{U}_1 \times \mathrm{SU}_8)$ over the symmetric space $\mathrm{E}_8/\mathrm{SO}_{16}$. Consider new left-invariant metrics on E_8 related to the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}$, i.e. $\langle \langle \cdot, \cdot \rangle \rangle' = v_1 \cdot B|_{\mathfrak{h}} + v_2 \cdot B|_{\mathfrak{n}}$ for some $v_1, v_2 \in \mathbb{R}_+$. For $v_1 = u_0 = u_1 = x_2$ and $v_2 = x_1 = x_3$ this metric coincides with the left-invariant metric $\langle \cdot, \cdot \rangle$. In a similar way, we get the relations $r_0 = r_1 = r_3$ and $r_2 = r_4$, which imply now $63A_{022} + 63A_{044} - A_{122} - A_{144} = 0$ and $3A_{033} - A_{111} - A_{133} = 0$. By combining with (4.6) we conclude. \square

Remark 4.5. An alternative way to compute A_{111} is given as follows. For $G \in \{\mathrm{G}_2(2), \mathrm{E}_8(8)\}$ consider the decomposition (4.1). Let K_1 be the connected Lie (sub)group generated by the simple ideal \mathfrak{k}_1 . Since $K_1 \subset G$, there exists a positive number $c > 0$ such that $B_{K_1} = c \cdot B_G$, where B_G denotes the negative of the Killing form of G . Let

$$a_{111} := \widetilde{\begin{bmatrix} 1 \\ 11 \end{bmatrix}}$$

the triple associated to K_1 with respect to B_{K_1} (as a compact simple Lie group). Then, by Lemma 2.2 we get $a_{111} = \dim \mathfrak{k}_1 =: d_1$. On the other hand, by the definition of A_{111} it is easy to see that $A_{111} = c \cdot a_{111}$ (for Lie groups with roots of the same length). For G_2 one has to notice that $\mathfrak{k}_1 \cong \mathfrak{su}(2)$ is generated by the long root α_1 , in particular $B_{\mathrm{G}_2}(\alpha_1, \alpha_1) = 3B_{\mathrm{G}_2}(\alpha_2, \alpha_2)$. Because $c = B_{\mathrm{SU}_2}/B_{\mathrm{G}_2} = 4/24$, it follows that $A_{111} = 3(c \cdot \dim \mathfrak{su}_2) = 3/2$. For $\mathrm{E}_8(8)$ it is $\mathfrak{k}_1 \cong \mathfrak{su}(8)$ and all the roots have the same length. In particular, $c = B_{\mathrm{SU}_8}/B_{\mathrm{E}_8} = 16/60$, hence $A_{111} = c \cdot \dim \mathfrak{su}_8 = 84/5$.

4.3. Naturally reductive metrics. For a Lie group $G \cong G(i_o)$ of Type $I_b(3)$, left-invariant metrics on $G \cong G(i_o)$ which are $\mathrm{Ad}(K)$ -invariant are given by (4.2).

Proposition 4.6. *If a left invariant metric $\langle \cdot, \cdot \rangle$ of the form (4.2) on $G \cong G(i_o)$ of Type $I_b(3)$ is naturally reductive with respect to $G \times L$ for some closed subgroup L of G , then one of the following holds:*

- (1) for $G = \mathrm{G}_2(2)$, $u_0 = x_2$, $x_1 = x_3$, and for $G = \mathrm{E}_8(8)$, $u_0 = u_1 = x_2$, $x_1 = x_3$ (2) $u_0 = u_1 = x_3$, $x_1 = x_2$ (3) $x_1 = x_2 = x_3$.

Conversely, if one of the conditions (1), (2), (3) holds, then the metric $\langle \cdot, \cdot \rangle$ of the form (4.2) is naturally reductive with respect to $G \times L$ for some closed subgroup L of G .

Proof. Let \mathfrak{l} be the Lie algebra of L . Then we have either $\mathfrak{l} \subset \mathfrak{k}$ or $\mathfrak{l} \not\subset \mathfrak{k}$. First we consider the case of $\mathfrak{l} \not\subset \mathfrak{k}$. Let \mathfrak{h} be the subalgebra of \mathfrak{g} generated by \mathfrak{l} and \mathfrak{k} . Since $\mathfrak{g} = \mathfrak{k}_0 \oplus \mathfrak{k}_1 \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3$ is an irreducible decomposition as $\mathrm{Ad}(K)$ -modules, we see that the Lie algebra \mathfrak{h} contains at least one of $\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3$. We first consider the case that \mathfrak{h} contains \mathfrak{p}_1 . Since $[\mathfrak{p}_1, \mathfrak{p}_1] \cap \mathfrak{p}_2 \neq \{0\}$, the space \mathfrak{h} contains \mathfrak{p}_2 . Notice also that $[\mathfrak{p}_1, \mathfrak{p}_2] \cap \mathfrak{p}_3 \neq \{0\}$, hence \mathfrak{h} contains \mathfrak{p}_3 . Thus we see that $\mathfrak{h} = \mathfrak{g}$ and the $\mathrm{Ad}(L)$ -invariant metric $\langle \cdot, \cdot \rangle$ of the form (4.2) is bi-invariant. Now, if \mathfrak{h} contains \mathfrak{p}_2 , then $\mathfrak{h} \supset \mathfrak{k} \oplus \mathfrak{p}_2$. If $\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{p}_2$, then $(\mathfrak{h}, \mathfrak{p}_1 \oplus \mathfrak{p}_3)$ is a symmetric pair. Thus the metric $\langle \cdot, \cdot \rangle$ of the form (4.2) satisfies $u_0 = x_2$, $x_1 = x_3$ for $G = \mathrm{G}_2(2)$ and $u_0 = u_1 = x_2$, $x_1 = x_3$ for $G = \mathrm{E}_8(8)$. If $\mathfrak{h} \neq \mathfrak{k} \oplus \mathfrak{p}_2$, we see that $\mathfrak{h} \cap \mathfrak{p}_1 \neq \{0\}$ or $\mathfrak{h} \cap \mathfrak{p}_3 \neq \{0\}$ and thus $\mathfrak{h} \supset \mathfrak{p}_1$ or $\mathfrak{h} \supset \mathfrak{p}_3$. Hence, we obtain $\mathfrak{h} = \mathfrak{g}$ and the $\mathrm{Ad}(L)$ -invariant metric $\langle \cdot, \cdot \rangle$ of the form (4.2) is bi-invariant. Next if \mathfrak{h} contains \mathfrak{p}_3 , then $\mathfrak{h} \supset \mathfrak{k} \oplus \mathfrak{p}_3$. If $\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{p}_3$, then \mathfrak{h} is a simple Lie algebra, in fact, for $G = \mathrm{G}_2(2)$ we see that $\mathfrak{h} = \mathfrak{su}_3$ and for $G = \mathrm{E}_8(8)$ we see that $\mathfrak{h} = \mathfrak{su}_9$, and $\mathfrak{p}_1 \oplus \mathfrak{p}_2$ is an irreducible $\mathrm{Ad}(H)$ -module. Thus the metric $\langle \cdot, \cdot \rangle$ of the form (4.2) satisfies $u_0 = u_1 = x_3$, $x_1 = x_2$. If $\mathfrak{h} \neq \mathfrak{k} \oplus \mathfrak{p}_3$, we see that $\mathfrak{h} \cap \mathfrak{p}_1 \neq \{0\}$ or $\mathfrak{h} \cap \mathfrak{p}_2 \neq \{0\}$ and thus $\mathfrak{h} \supset \mathfrak{p}_1$ or $\mathfrak{h} \supset \mathfrak{p}_2$. Hence, we obtain $\mathfrak{h} = \mathfrak{g}$ and the $\mathrm{Ad}(L)$ -invariant metric $\langle \cdot, \cdot \rangle$ of the form (4.2) is bi-invariant.

We proceed with the case $\mathfrak{l} \subset \mathfrak{k}$. Because the orthogonal complement \mathfrak{l}^\perp of \mathfrak{l} with respect to B contains the orthogonal complement \mathfrak{k}^\perp of \mathfrak{k} , it follows that $\mathfrak{l}^\perp \supset \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3$. Since the invariant metric $\langle \cdot, \cdot \rangle$ is naturally reductive with respect to $G \times L$, we conclude that $x_1 = x_2 = x_3$ by Theorem 2.1.

Conversely, if the conditions (1) holds, then Theorem 2.1 states that the metric $\langle \cdot, \cdot \rangle$ given by (4.2) is naturally reductive with respect to $G \times L$, where $\mathfrak{l} = \mathfrak{k} \oplus \mathfrak{p}_2$. If the condition (2) holds, then the metric $\langle \cdot, \cdot \rangle$ given by (4.2) is naturally reductive with respect to $G \times L$ where $\mathfrak{l} = \mathfrak{k} \oplus \mathfrak{p}_3$. Finally, if the condition (3) holds, then the metric defined by (4.2) is naturally reductive with respect to $G \times K$. \square

4.4. Einstein metrics. Due to Lemma 4.4 and Remark 4.1, Corollary 4.3 determines now explicitly the Ricci tensor $\mathrm{Ric}_{\langle \cdot, \cdot \rangle}$ of the Lie groups $(\mathrm{G}_2(2), \langle \cdot, \cdot \rangle)$ and $(\mathrm{E}_8(8), \langle \cdot, \cdot \rangle)$. Recall that the homogeneous Einstein equation for the left-invariant metric $\langle \cdot, \cdot \rangle$ is given by

$$\{r_0 - r_1 = 0, r_1 - r_2 = 0, r_2 - r_3 = 0, r_3 - r_4 = 0\}.$$

Case of $G_2(2)$

We normalize the metric by setting $x_3 = 1$. Then, we see that the homogeneous Einstein equation is equivalent to the following system of equations:

$$(4.7) \quad \begin{cases} f_1 &= 9u_0u_1x_1^2x_2^2 + 2u_0u_1x_1^2 + u_0u_1x_2^2 - 3u_1^2x_1^2x_2^2 - 3u_1^2x_2^2 - 6x_1^2x_2^2 = 0, \\ f_2 &= u_0u_1x_2 + 6u_1^2x_1^2x_2 + 15u_1^2x_2 - 6u_1x_1^3 + 6u_1x_1x_2^2 - 48u_1x_1x_2 + 6u_1x_1 \\ &\quad + 8u_1x_2^2 + 12x_1^2x_2 = 0, \\ f_3 &= 4u_0x_1^2 - u_0x_2^2 - 9u_1x_2^2 + 18x_1^3x_2 - 32x_1^2x_2 - 18x_1x_2^3 + 48x_1x_2^2 \\ &\quad + 6x_1x_2 - 16x_2^3 = 0, \\ f_4 &= 9u_0x_1^2x_2^2 - 4u_0x_1^2 + 9u_1x_1^2x_2^2 - 6x_1^3x_2 - 48x_1^2x_2^2 + 32x_1^2x_2 \\ &\quad + 18x_1x_2^3 - 18x_1x_2 + 8x_2^3 = 0. \end{cases}$$

Consider the polynomial ring $R = \mathbb{Q}[z, u_0, u_1, x_1, x_2]$ and the ideal I , generated by polynomials $\{z u_0 u_1 x_1 x_2 - 1, f_1, f_2, f_3, f_4\}$. We take a lexicographic ordering $>$, with $z > u_0 > u_1 > x_2 > x_1$ for a monomial ordering on R . Then, by the aid of computer, we see that a Gröbner basis for the ideal I contains a polynomial of x_1 given by $(x_1 - 1)(9x_1 - 11)h_1(x_1)$, where $h_1(x_1)$ is a polynomial of degree 39 of the form

$$\begin{aligned} h_1(x_1) = & 578531204393508729x_1^{39} - 2907419178698478477x_1^{38} + 9728488450924774839x_1^{37} \\ & - 25248552448377295323x_1^{36} + 56466222208555751172x_1^{35} - 112561770533625695268x_1^{34} \\ & + 198560662542278445420x_1^{33} - 303749957092666314564x_1^{32} + 403108239570614919684x_1^{31} \\ & - 461651040693288248940x_1^{30} + 457287888650310982692x_1^{29} - 385901524001421918252x_1^{28} \\ & + 263698285244084996724x_1^{27} - 121027010977188296460x_1^{26} - 3723246665533789740x_1^{25} \\ & + 82231517231748069876x_1^{24} - 102191761959692380074x_1^{23} + 82264683344411071386x_1^{22} \\ & - 47064365277272324622x_1^{21} + 17165576727452434302x_1^{20} - 1543691819299617324x_1^{19} \\ & - 4724193236441019084x_1^{18} + 4420128602614576596x_1^{17} - 3155137453029513948x_1^{16} \\ & + 1592256867663356196x_1^{15} - 768600370620963068x_1^{14} + 294461889742168084x_1^{13} \\ & - 112323449859022284x_1^{12} + 34778832154783148x_1^{11} - 10968398600557556x_1^{10} \\ & + 2775906314750316x_1^9 - 734820705350612x_1^8 + 149714619756553x_1^7 \\ & - 33449638312869x_1^6 + 5237027887391x_1^5 - 997188227243x_1^4 \\ & + 107164942344x_1^3 - 17880627984x_1^2 + 968584320x_1 - 150264576. \end{aligned}$$

We also remark that in the Gröbner basis, u_0, u_1, x_2 are given by polynomials of degree 40 of x_1 with coefficients of rational numbers. Solving $h_1(x_1) = 0$ numerically, we get only one solution, which is given approximately by $x_1 \approx 0.93245951$. Further, we see that a solution of the system of equations $\{f_1 = 0, f_2 = 0, f_3 = 0, f_4 = 0, h_1(x_1) = 0\}$ has the form by

$$\{u_0 \approx 1.0851961, u_1 \approx 0.69929486, x_1 \approx 0.93245951, x_2 \approx 1.0225069\}.$$

Due to Proposition 4.6, we conclude that this solution induces a non-naturally reductive Einstein metric.

For $x_1 = 11/9$, the system $\{f_1 = 0, f_2 = 0, f_3 = 0, f_4 = 0\}$ has a solution, given by

$$\{u_0 = 1, u_1 = 1, x_1 = 11/9, x_2 = 11/9\}.$$

For $x_1 = 1$, we get $u_0 = x_2$, $(x_2 - 1)(875x_2^3 - 1165x_2^2 + 250x_2 - 14) = 0$ and $u_1 = (1750x_2^3 - 4080x_2^2 + 2585x_2 - 192)/63$. Thus, we get solutions of the system of equations $\{f_1 = 0, f_2 = 0, f_3 = 0, f_4 = 0\}$, namely

$$\begin{aligned} & \{u_0 \approx 0.095267235, u_1 \approx 0.29761039, x_1 = 1, x_2 \approx 0.095267235\}, \\ & \{u_0 \approx 0.15539816, u_1 \approx 1.8689705, x_1 = 1, x_2 \approx 0.15539816\}, \\ & \{u_0 \approx 1.0807632, u_1 \approx 0.71913340, x_1 = 1, x_2 \approx 1.0807632\} \end{aligned}$$

and $u_0 = u_1 = x_1 = x_2 = 1$. By Proposition 4.6, one can deduce that these values give rise to naturally reductive Einstein metrics.

Case of $E_8(8)$

For a normalization of the metric $x_3 = 1$, the homogeneous Einstein equation is equivalent to the following system of equations:

$$(4.8) \quad \begin{cases} g_1 = 14u_0u_1x_1^2 - 6u_1^2x_1^2 + 7u_0u_1x_2^2 - 15u_1^2x_2^2 - 8x_1^2x_2^2 + 9u_0u_1x_1^2x_2^2 - u_1^2x_1^2x_2^2 = 0, \\ g_2 = 48u_1^2x_1^2 + 24u_1x_1x_2 - 24u_1x_1^3x_2 + u_0u_1x_2^2 + 255u_1^2x_2^2 - 480u_1x_1x_2^2 + 64x_1^2x_2^2 \\ + 8u_1^2x_1^2x_2^2 + 80u_1x_2^3 + 24u_1x_1x_2^3 = 0, \\ g_3 = 14u_0x_1^2 + 108u_1x_1^2 + 24x_1x_2 - 320x_1^2x_2 + 72x_1^3x_2 - u_0x_2^2 - 135u_1x_2^2 + 480x_1x_2^2 \\ - 160x_2^3 - 72x_1x_2^3 = 0, \\ g_4 = -4u_0x_1^2 - 108u_1x_1^2 - 216x_1x_2 + 320x_1^2x_2 + 120x_1^3x_2 - 480x_1^2x_2^2 + 9u_0x_1^2x_2^2 \\ + 63u_1x_1^2x_2^2 + 80x_2^3 + 216x_1x_2^3 = 0. \end{cases}$$

We consider the polynomial ring $R = \mathbb{Q}[z, u_0, u_1, x_1, x_2]$ and an ideal I , generated by polynomials $\{z u_0 u_1 x_1 x_2 - 1, g_1, g_2, g_3, g_4\}$. We choose the lexicographic ordering $>$, with $z > u_0 > u_1 > x_2 > x_1$ for a monomial ordering on R . Then, a Gröbner basis for the ideal I contains a polynomial of x_1 , given by $(x_1 - 1)(9x_1 - 41)k_1(x_1)$, where $k_1(x_1)$ is a polynomial of degree 49 explicitly defined as follows:

$$\begin{aligned} k_1(x_1) = & 78627620134518984869670299619225x_1^{49} - 1879012156849489779707699697741525x_1^{48} \\ & + 24958517859683233851773742581453130x_1^{47} - 226345877119100379348478414653942870x_1^{46} \\ & + 1513185548162200779194248616933419791x_1^{45} - 7767087422550952023317555159123235339x_1^{44} \\ & + 30802377992721718905775487881405527156x_1^{43} - 91469740820715969757364654368958734704x_1^{42} \\ & + 190873255583958103333447847157388070763x_1^{41} - 251450263614444457947297352429222170207x_1^{40} \\ & + 174697300086572957060190519165619552914x_1^{39} - 348342277415980322916046484535937167846x_1^{38} \\ & + 3002677791406351946122412362593386235357x_1^{37} - 14163273159456167553892516550563377195873x_1^{36} \\ & + 44027812533760332962711411379822436468296x_1^{35} - 103359433218793762175332925928788224767564x_1^{34} \\ & + 195341684679658672333566251584926273888618x_1^{33} - 305149823721731252899197653136452852072562x_1^{32} \\ & + 393503106465236359948116939782270035587444x_1^{31} - 405394724912565051721521587945499542805996x_1^{30} \\ & + 272987735753676571252110160502472211820742x_1^{29} + 67527266567097795889985999048107668548322x_1^{28} \\ & - 648798451413297437284550685137354441143464x_1^{27} + 1406965042656066428973970557643766067268176x_1^{26} \\ & - 2175743101927487127818021138090846891067546x_1^{25} + 2652679256280234163460510381340166807164594x_1^{24} \\ & - 2558905970930356615952062891716100348759020x_1^{23} + 1819840234201169247432197581725018033034180x_1^{22} \\ & - 573110366219719484295636119428516440072790x_1^{21} - 898192008904623391867803363166579084394930x_1^{20} \\ & + 2196777571676601155406190148800336652452480x_1^{19} - 3047573028495615879932513597313365789288840x_1^{18} \\ & + 3388852959467640315510569932345608419421165x_1^{17} - 3273630300427318361235238941319922925655545x_1^{16} \\ & + 2823340833679036161326179394502997732627650x_1^{15} - 2191869205433840326556829886605796355495550x_1^{14} \\ & + 1538080739419688790208901918515707596638875x_1^{13} - 976760862170802820093764370749377583728775x_1^{12} \\ & + 563675702099787379912980508814827349226900x_1^{11} - 295654068049591813276291328141962741142400x_1^{10} \\ & + 141495683505513028215917983599798929865375x_1^9 - 61509598161488435381920159529683875449475x_1^8 \\ & + 24312853879710780673048343979366706124250x_1^7 - 8664729466196887937652173231674074174750x_1^6 \\ & + 277023499900055864676202239898046895625x_1^5 - 786852533021232630512967801167059768125x_1^4 \\ & + 19473327136191111513857132379687575000x_1^3 - 41207441950449158811069657833488687500x_1^2 \\ & + 6741277041530521149993243000562500000x_1 - 859782169024171126981444722656250000. \end{aligned}$$

Notice that in the Gröbner basis, u_0, u_1, x_2 are given by polynomials of degree 50 of x_1 with coefficients of rational numbers. Solving $k_1(x_1) = 0$ numerically, we get three positive and two negative solutions, which are given approximately by $x_1 \approx 0.46131382, x_1 \approx 0.91172474, x_1 \approx 4.0130840$ and $x_1 \approx -1.2146356, x_1 \approx -1.1542138$. Moreover, solutions of the system $\{g_1 = 0, g_2 = 0, g_3 = 0, g_4 = 0\}$ have the approximate form

$$\begin{aligned} & \{u_0 \approx 1.0767925, u_1 \approx 0.12842350, x_1 \approx 0.46131382, x_2 \approx 0.73659849\}, \\ & \{u_0 \approx 0.77844700, u_1 \approx 0.17409566, x_1 \approx 0.91172474, x_2 \approx 0.52532563\}, \\ & \{u_0 \approx 1.1022316, u_1 \approx 0.85179391, x_1 \approx 4.0130840, x_2 \approx 4.0222155\} \end{aligned}$$

and

$$\begin{aligned} & \{u_0 \approx -1.3411877, u_1 \approx -0.75642675, x_1 \approx -1.2146356, x_2 \approx -4.9166783\}, \\ & \{u_0 \approx -1.4503818, u_1 \approx -0.54000582, x_1 \approx -1.1542138, x_2 \approx -4.7370866\}. \end{aligned}$$

Thus, we obtain three Einstein metrics which are non-naturally reductive, by Proposition 4.6. After computing the related scale invariants (cf. [AC, Section 7]) we deduce that these metrics are non-isometric each other.

For $x_1 = 9/41$, the system of equation $\{f_1 = 0, f_2 = 0, f_3 = 0, f_4 = 0\}$ has a solution, given by

$$\{u_0 = 1, u_1 = 1, x_1 = 9/41, x_2 = 9/41\}.$$

For $x_1 = 1$, we see that $u_0 = u_1 = x_2 = 7/23$ and $u_0 = u_1 = x_1 = x_2 = 1$. Due to Proposition 4.6, these solutions define naturally reductive Einstein metrics.

5. LEFT-INVARIANT NON-NATURALLY REDUCTIVE EINSTEIN METRICS ON LIE GROUPS OF TYPE $II_b(3)$

Let $G \cong G(i_o)$ be a compact connected Lie groups of Type $II_b(3)$. Then G is isometric to $F_4(2)$, $E_7(3)$, $E_8(2)$ or $E_7(5)$. Let $\mathfrak{g} = T_e G$ be the corresponding Lie algebra.

5.1. The Ricci tensor. For a Lie group $G \cong G(i_o)$ of Type $II_b(3)$ consider the orthogonal decomposition

$$(5.1) \quad \mathfrak{g} = \mathfrak{k}_0 \oplus \mathfrak{k}_1 \oplus \mathfrak{k}_2 \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3 = \mathfrak{m}_0 \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3 \oplus \mathfrak{m}_4 \oplus \mathfrak{m}_5.$$

This is a reductive decomposition of \mathfrak{g} of the form (2.1) and a left-invariant metric on $G \cong G(i_o)$ is given by

$$(5.2) \quad \begin{aligned} \langle \cdot, \cdot \rangle &= u_0 \cdot B|_{\mathfrak{k}_0} + u_1 \cdot B|_{\mathfrak{k}_1} + u_2 \cdot B|_{\mathfrak{k}_2} + x_1 \cdot B|_{\mathfrak{p}_1} + x_2 \cdot B|_{\mathfrak{p}_2} + x_3 \cdot B|_{\mathfrak{p}_3} \\ &= y_0 \cdot B|_{\mathfrak{m}_0} + y_1 \cdot B|_{\mathfrak{m}_1} + y_2 \cdot B|_{\mathfrak{m}_2} + y_3 \cdot B|_{\mathfrak{m}_3} + y_4 \cdot B|_{\mathfrak{m}_4} + y_5 \cdot B|_{\mathfrak{m}_5} \end{aligned}$$

for some positive numbers $u_p, x_i, y_j \in \mathbb{R}_+$. Thus, a G -invariant metric on $M = G/K$ is of the form

$$(\cdot, \cdot) = x_1 \cdot B|_{\mathfrak{p}_1} + x_2 \cdot B|_{\mathfrak{p}_2} + x_3 \cdot B|_{\mathfrak{p}_3} = y_3 \cdot B|_{\mathfrak{m}_3} + y_4 \cdot B|_{\mathfrak{m}_4} + y_5 \cdot B|_{\mathfrak{m}_5}.$$

For a Lie group $G \cong G(i_o)$ of Type $II_b(3)$ in Table 3 we state the subalgebras \mathfrak{k}_i , the dimensions $d_i := \dim_{\mathbb{R}} \mathfrak{m}_i$ for $i = 1, \dots, 5$ ($d_0 = 1$) and the vanishing or not of the triple A_{144} , which plays an essential role.

Table 3. The simple ideals \mathfrak{k}_i , the dimensions d_i , and the vanishing of A_{144}

$G(i_o)$	\mathfrak{k}_1	\mathfrak{k}_2	d_1	d_2	d_3	d_4	d_5	$A_{144} = 0$
$F_4(2)$	\mathfrak{su}_2	\mathfrak{su}_3	3	8	24	12	4	✓
$E_7(5)$	\mathfrak{su}_2	\mathfrak{su}_6	3	35	60	30	4	✓
$E_8(2)$	\mathfrak{su}_2	\mathfrak{e}_6	3	78	108	54	4	✓
$E_7(3)$	\mathfrak{su}_5	\mathfrak{su}_3	24	8	60	30	10	$A_{144} \neq 0$

Proposition 5.1. *For the reductive decomposition (5.1) associated to the compact simple Lie groups $F_4(2)$, $E_7(5)$, $E_8(2)$ and for the left-invariant metric given by (5.2), the non-zero structure constants A_{ijk} ($0 \leq i, j, k \leq 5$) are the following (and their symmetries): A_{033} , A_{044} , A_{055} , A_{111} , A_{133} , A_{155} , A_{222} , A_{233} , A_{244} , A_{334} , and A_{345} . This also holds for $E_7(3)$, but in this case one has in addition $A_{144} \neq 0$.*

Proof. Similarly with Proposition 4.2, the triples A_{334}, A_{345} are non-zero because $M = G/K$ is such that $b_2(M) = 1$ with $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3$. From the other cases we describe these which are less obvious.

Consider the decomposition (5.1) of the Lie algebra \mathfrak{g} and let $\mathfrak{t} \subset \mathfrak{g}$ be a maximal abelian subalgebra. Let $\mathfrak{g}^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} \oplus \sum_{\alpha \in R} \mathfrak{g}^{\alpha}$ the root space decomposition of the complexification $\mathfrak{g}^{\mathbb{C}} := \mathfrak{g} \otimes \mathbb{C} = \mathfrak{t}^{\mathbb{C}} \oplus \mathfrak{p}_1^{\mathbb{C}} \oplus \mathfrak{p}_2^{\mathbb{C}} \oplus \mathfrak{p}_3^{\mathbb{C}}$ with respect to the Cartan subalgebra (CSA) $\mathfrak{t}^{\mathbb{C}} \subset \mathfrak{g}^{\mathbb{C}}$. Next we shall identify roots $\alpha \in R$ with vectors $H_{\alpha} \in \sqrt{-1}\mathfrak{t}$, defined by $\alpha(H) = B(H_{\alpha}, H)$, $\forall H \in \mathfrak{t}^{\mathbb{C}}$. Choose a Weyl basis of root vectors $\{E_{\alpha} \in \mathfrak{g}^{\alpha} : \alpha \in R\}$ and let $\Pi = \{\alpha_1, \dots, \alpha_{\ell}\}$ ($\ell = \dim \mathfrak{t}$) be the fixed fundamental basis and R^+ the associated positive roots. Then, there exists a subset $\Pi_K \subset \Pi$ such that $R_K = R \cap \langle \Pi_K \rangle$ be the root system of the (semi-simple part) of the reductive Lie algebra $\mathfrak{t}^{\mathbb{C}} = Z(\mathfrak{t}^{\mathbb{C}}) \oplus \mathfrak{t}_{ss}^{\mathbb{C}}$, where $\langle \Pi_K \rangle$ is the subspace of $\sqrt{-1}\mathfrak{t}$ generated by Π_K with integer coefficients. Similarly, we write $R_K^+ := R^+ \cap \langle \Pi_K \rangle$ for the corresponding positive roots. Due to reductive decomposition (5.1), R_K splits into two root subsystems, say R_{K_1}, R_{K_2} , which can be identified with the root systems of $\mathfrak{k}_i^{\mathbb{C}} = \mathfrak{k}_i \otimes \mathbb{C}$ ($i = 1, 2$). Hence we get $R_K = R_{K_1} \sqcup R_{K_2}$, $\Pi_K = \Pi_{K_1} \sqcup \Pi_{K_2}$, e.t.c., and we decompose $\mathfrak{t}_{ss}^{\mathbb{C}}$ as follows:

$$\mathfrak{t}_{ss}^{\mathbb{C}} := [\mathfrak{t}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}}] = \sum_{\alpha \in \Pi_{K_1}} \mathbb{C}H_{\alpha} \oplus \sum_{\beta \in \Pi_{K_2}} \mathbb{C}H_{\beta} \oplus \sum_{\alpha \in R_{K_1}} \mathfrak{g}^{\alpha} \oplus \sum_{\beta \in R_{K_2}} \mathfrak{g}^{\beta} = \mathfrak{t}' \oplus \sum_{\alpha \in R_{K_1}} \mathfrak{g}^{\alpha} \oplus \sum_{\beta \in R_{K_2}} \mathfrak{g}^{\beta}.$$

Here $\mathfrak{t}' := \{\sum_{\alpha \in \Pi_{K_1}} \mathbb{C}H_{\alpha} \oplus \sum_{\beta \in \Pi_{K_2}} \mathbb{C}H_{\beta}\} \subset \mathfrak{t}^{\mathbb{C}}$ is a CSA of $\mathfrak{t}_{ss}^{\mathbb{C}}$. Set now $\mathfrak{h} := \{H \in \mathfrak{t} : (H, \Pi_K) = 0\}$; this is a real form of the centre $Z(\mathfrak{t}^{\mathbb{C}}) \cong \mathfrak{k}_0^{\mathbb{C}}$ of $\mathfrak{t}^{\mathbb{C}}$. Since $\mathfrak{h} \subset \mathfrak{t} \subset \mathfrak{k}$, we write $\mathfrak{t} = \mathfrak{h} \oplus \mathfrak{h}^{\perp}$ with

$$\mathfrak{h}^{\perp} \cong \text{span}\{\sqrt{-1}H_{\alpha} : \alpha \in \Pi_{K_1}\} \oplus \text{span}\{\sqrt{-1}H_{\beta} : \beta \in \Pi_{K_2}\} := \mathfrak{h}_1^{\perp} \oplus \mathfrak{h}_2^{\perp}.$$

Hence the CSA $\mathfrak{t}' \subset \mathfrak{t}^{\mathbb{C}} \subset \mathfrak{k}^{\mathbb{C}}$ is just the complexification of \mathfrak{h}^{\perp} , i.e. $\mathfrak{t}' = \mathfrak{h}^{\perp} \oplus \sqrt{-1}\mathfrak{h}^{\perp}$. Then, one has that

$$\dim_{\mathbb{R}} \mathfrak{h}^{\perp} = \dim_{\mathbb{C}} \mathfrak{t}' = |\Pi_K| = |\Pi_{K_1}| + |\Pi_{K_2}| = \dim_{\mathbb{R}} \mathfrak{h}_1^{\perp} + \dim_{\mathbb{R}} \mathfrak{h}_2^{\perp} \Rightarrow \dim_{\mathbb{R}} \mathfrak{h} = \ell - |\Pi_K| = 1.$$

Thus $\sqrt{-1}\mathfrak{h} \cong \mathfrak{k}_0 = \mathfrak{u}_1$. For $F_4(2)$, $E_7(5)$ and $E_8(2)$ it is $\mathfrak{k}_1 \cong \mathfrak{su}_2$ with corresponding root system $R_{K_1} = \{\pm\alpha_1\}$, $R_{K_1} = \{\pm\alpha_1\}$ and $R_{K_1} = \{\pm\alpha_6\}$, respectively. If $G = G(i_o)$ is $E_7(3)$, then $\mathfrak{k}_1 \cong \mathfrak{su}_5$ and R_{K_1} is more

complicated (see below). In the following table, for any Lie group $G = G(i_o)$ of Type $II_b(3)$ we summarize the data encoded by the decomposition $\mathfrak{t} = \mathfrak{h} \oplus \mathfrak{h}_1^\perp \oplus \mathfrak{h}_2^\perp$.

	$\dim \mathfrak{t}$	Π_{K_1}	$\dim \mathfrak{h}_1^\perp$	Π_{K_2}	$\dim \mathfrak{h}_2^\perp$	$\Pi_M = \{\alpha_{i_o}\}$
$F_4(2)$	4	$\{\alpha_1\}$	1	$\{\alpha_3, \alpha_4\}$	2	α_2
$E_7(5)$	7	$\{\alpha_6\}$	1	$\{\alpha_1, \dots, \alpha_4, \alpha_7\}$	5	α_5
$E_8(2)$	8	$\{\alpha_1\}$	1	$\{\alpha_3, \dots, \alpha_8\}$	6	α_2
$E_7(3)$	7	$\{\alpha_4, \dots, \alpha_7\}$	4	$\{\alpha_1, \alpha_2\}$	2	α_3

Now, \mathfrak{t} is a common maximal abelian subalgebra of $\mathfrak{k} \subset \mathfrak{g}$. Hence $\mathfrak{k} = \mathfrak{u}_1 \oplus \mathfrak{k}_1 \oplus \mathfrak{k}_2 = \mathfrak{t} \oplus \sum_{\alpha \in R_K^+} \{\mathbb{R}A_\alpha \oplus \mathbb{R}B_\alpha\}$, where $A_\alpha := (E_\alpha + E_{-\alpha})$ and $B_\alpha := \sqrt{-1}(E_\alpha - E_{-\alpha})$. The simple ideals $\mathfrak{k}_1, \mathfrak{k}_2$ can be viewed as

$$\mathfrak{k}_1 = \sum_{\alpha \in \Pi_{K_1}} \mathbb{R}H_\alpha \oplus \sum_{\alpha \in R_{K_1}^+} \{\mathbb{R}A_\alpha \oplus \mathbb{R}B_\alpha\} = \mathfrak{h}_1^\perp \oplus \sum_{\alpha \in R_{K_1}^+} \{\mathbb{R}A_\alpha \oplus \mathbb{R}B_\alpha\}, \quad (*)$$

$$\mathfrak{k}_2 = \sum_{\beta \in \Pi_{K_2}} \mathbb{R}H_\beta \oplus \sum_{\beta \in R_{K_2}^+} \{\mathbb{R}A_\beta \oplus \mathbb{R}B_\beta\} = \mathfrak{h}_2^\perp \oplus \sum_{\beta \in R_{K_2}^+} \{\mathbb{R}A_\beta \oplus \mathbb{R}B_\beta\}. \quad (**)$$

Let $R_M^+ := R^+ \setminus (R_{K_1}^+ \sqcup R_{K_2}^+)$ be the complementary roots of $M = G/K$. Because $\text{ht}(\alpha_{i_o}) = 3$, this set splits into 3 subsets $R_M^+ = R_1^+ \sqcup R_2^+ \sqcup R_3^+$, given by $R_t^+ := \{\alpha = \sum_{j=1}^4 c_j \alpha_j \in R_M^+ : c_{i_o} = t, 1 \leq t \leq 3\}$. Then

$$(5.3) \quad \mathfrak{m}_3 \cong \mathfrak{p}_1 = \sum_{\alpha \in R_1^+} \{\mathbb{R}A_\alpha \oplus \mathbb{R}B_\alpha\}, \quad \mathfrak{m}_4 \cong \mathfrak{p}_2 = \sum_{\alpha \in R_2^+} \{\mathbb{R}A_\alpha \oplus \mathbb{R}B_\alpha\}, \quad \mathfrak{m}_5 \cong \mathfrak{p}_3 = \sum_{\alpha \in R_3^+} \{\mathbb{R}A_\alpha \oplus \mathbb{R}B_\alpha\}.$$

1st way. Fix one of the Lie groups $F_4(2)$, $E_7(5)$ and $E_8(2)$. A method to prove that $A_{144} = 0$ but $A_{133} \neq 0$, $A_{155} \neq 0$, is via the inclusions $[\mathfrak{k}_1, \mathfrak{m}_3] \subset \mathfrak{m}_3$, $[\mathfrak{k}_1, \mathfrak{m}_5] \subset \mathfrak{m}_5$ and $[\mathfrak{k}_1, \mathfrak{m}_4] = 0$. These can be obtained directly by considering root vectors associated to roots of $R_{K_1}^+$ and R_M^+ and combining (*) and (5.3). In the same way and by using now (**) and (5.3) we get $[\mathfrak{k}_2, \mathfrak{m}_5] = 0$, which implies that $A_{255} = 0$. Of course, this method applies also for $E_7(3)$; for this Lie group the inclusion $[\mathfrak{k}_1, \mathfrak{m}_4] \subset \mathfrak{m}_4$ holds, so $A_{144} \neq 0$.

2nd way. An alternative way to examine the behaviour of A_{144} is based on the orthogonality of roots. Let us denote the unique simple root belonging in Π_{K_1} by ϕ (recall that $|\Pi_{K_1}| = 1$ for $F_4(2)$, $E_7(5)$ and $E_8(2)$). Consider some complementary root $\alpha \in R_2^+$ associated to $\mathfrak{p}_2 \cong \mathfrak{m}_4$. In terms of simple roots, α may be expressed by $\alpha = \phi + 2\alpha_{i_o} + \sum_k c_k \alpha_k$ with $(i_o < k \leq \ell)$, or $\alpha = \sum_k c_k \alpha_k + 2\alpha_{i_o} + \phi$ with $(1 \leq k < i_o)$. In full details:

$$F_4(2) : \alpha = \phi + 2\alpha_2 + \sum_{k=3}^4 c_k \alpha_k, \quad E_7(5) : \alpha = \sum_{k=1}^4 c_k \alpha_k + c_7 \alpha_7 + 2\alpha_5 + \phi, \quad E_8(2) : \alpha = \phi + 2\alpha_2 + \sum_{k=3}^8 c_k \alpha_k.$$

The fact that the simple root $\phi \in \Pi_K \equiv R_K^+$ appears in the expression of $\alpha \in R_2^+$ always with coefficient 1, can be straightforward checked by the expressions of positive roots in terms of simple roots. Since ϕ is connected to $-\tilde{\alpha}$ it is $(\phi, \tilde{\alpha}) \neq 0$ (in general $(\alpha_i, \tilde{\alpha}) \geq 0$). By assuming now that $\alpha = \phi + 2\alpha_{i_o} + \sum_k c_k \alpha_k$ (the other case is treated similarly) we get that

$$\frac{2(\phi + 2\alpha_{i_o} + \sum_k c_k \alpha_k, \phi)}{(\phi, \phi)} = \frac{2(\phi + 2\alpha_{i_o}, \phi)}{(\phi, \phi)} = \frac{2(\phi, \phi)}{(\phi, \phi)} + \frac{2(2\alpha_{i_o}, \phi)}{(\phi, \phi)} = 1 - 1 = 0,$$

since $2(\alpha_{i_o}, \phi) = -(\phi, \phi)$. This shows that the orthogonality of the roots α and ϕ , and since ϕ spans $R_{K_1}^+$, by the definition of A_{ijk} we conclude that $A_{144} = 0$. Let us illustrate the computations shortly for $F_4(2)$. Let $\Pi = \{\alpha = e_2 - e_3, \alpha_2 = e_3 - e_4, \alpha_3 = e_4, \alpha_4 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4)\}$ be the fixed fundamental basis, with $\tilde{\alpha} = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$. Then, the Cartan matrix is given by

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \Rightarrow \frac{2(\alpha_1 + 2\alpha_2 + c_3\alpha_3 + c_4\alpha_4, \alpha_1)}{(\alpha_1, \alpha_1)} = \frac{2(\alpha_1, \alpha_1)}{(\alpha_1, \alpha_1)} + \frac{2(2\alpha_2, \alpha_1)}{(\alpha_1, \alpha_1)} = 1 - 1 = 0.$$

We explain now why $A_{144} \neq 0$ for $E_7(3)$. Recall that $\Pi_{K_1} = \{\alpha_4, \dots, \alpha_7\}$, with result $R_{K_1} \cong R_{\text{SU}_5}$ and $R_{K_1}^+ = \Pi_K \sqcup \{\alpha_4 + \alpha_5, \alpha_4 + \alpha_5 + \alpha_6, \alpha_4 + \alpha_7, \alpha_5 + \alpha_6, \alpha_4 + \alpha_5 + \alpha_7, \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7\}$. For convenience, we present the set R_2^+ in terms of simple roots (we state only the coefficients)

$$R_2^+ = \left\{ \begin{pmatrix} 0, 1, 2, 2, 1, 0, 1 \end{pmatrix}, \begin{pmatrix} 0, 1, 2, 2, 1, 1, 1 \end{pmatrix}, \begin{pmatrix} 0, 1, 2, 2, 2, 1, 1 \end{pmatrix}, \begin{pmatrix} 0, 1, 2, 3, 2, 1, 1 \end{pmatrix}, \begin{pmatrix} 1, 1, 2, 2, 1, 0, 1 \end{pmatrix}, \right. \\ \left. \begin{pmatrix} 1, 1, 2, 2, 1, 1, 1 \end{pmatrix}, \begin{pmatrix} 1, 1, 2, 2, 2, 1, 1 \end{pmatrix}, \begin{pmatrix} 1, 1, 2, 3, 2, 1, 1 \end{pmatrix}, \begin{pmatrix} 1, 2, 2, 2, 1, 0, 1 \end{pmatrix}, \begin{pmatrix} 1, 2, 2, 2, 1, 1, 1 \end{pmatrix}, \right. \\ \left. \begin{pmatrix} 1, 2, 2, 2, 2, 1, 1 \end{pmatrix}, \begin{pmatrix} 1, 2, 2, 3, 2, 1, 1 \end{pmatrix}, \begin{pmatrix} 0, 1, 2, 3, 2, 1, 2 \end{pmatrix}, \begin{pmatrix} 1, 1, 2, 3, 2, 1, 2 \end{pmatrix}, \begin{pmatrix} 1, 2, 2, 3, 2, 1, 2 \end{pmatrix} \right\}.$$

Choose for example $\alpha := \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7 \in R_2^+$ and $\phi := \alpha_4 + \alpha_7 \in R_{K_1}^+$. Then, $[E_\alpha, E_\phi] = N_{\alpha\phi}E_{\alpha+\phi} \neq 0$, since $\alpha + \phi = \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + 2\alpha_7 \in R_2^+$, i.e. $[E_\alpha, E_\phi] \in \mathfrak{m}_4$ and hence $A_{144} \neq 0$. This can be verified by the orthogonality of roots as well; since the fixed basis of simple roots of E_7 is such that $(\alpha_i, \alpha_i) = 2$ and $(\alpha_1, \alpha_2) = (\alpha_2, \alpha_3) = (\alpha_3, \alpha_4) = (\alpha_4, \alpha_5) = (\alpha_4, \alpha_7) = (\alpha_5, \alpha_6) = -1$, it follows that

$$\frac{2(\alpha, \phi)}{(\phi, \phi)} = -3/2 \neq 0.$$

We finish the proof with a short remark about A_{225} . This case can be treated by similar methods as above, however it occurs in a faster way via the painted Dynkin diagram associated to a Lie group $G = G(i_o)$ of Type $II_b(3)$. For such a group and the reductive decomposition of its Lie algebra \mathfrak{g} given by (5.1), observe that the Dynkin diagram of \mathfrak{k}_2 is not connected with $-\tilde{\alpha}$. Hence $[\mathfrak{k}_2, E_{\tilde{\alpha}}] = 0$, and since $\tilde{\alpha} \in R_3^+$, i.e. $E_{\tilde{\alpha}} \in \mathfrak{p}_3 \cong \mathfrak{m}_5$, it follows that $A_{225} = 0$. \square

Now, an easy application of Lemma 2.2 gives that

Corollary 5.2. *On $(E_7(3), \langle \cdot, \cdot \rangle)$, the components r_i of the Ricci tensor $\text{Ric}_{\langle \cdot, \cdot \rangle}$ associated to the left-invariant metric $\langle \cdot, \cdot \rangle$ given by (5.2), are described as follows*

$$\left\{ \begin{array}{l} r_0 = \frac{u_0}{4d_0} \left(\frac{A_{033}}{x_1^2} + \frac{A_{044}}{x_2^2} + \frac{A_{055}}{x_3^2} \right), \quad r_1 = \frac{A_{111}}{4d_1} \cdot \frac{1}{u_1} + \frac{u_1}{4d_1} \left(\frac{A_{133}}{x_1^2} + \frac{A_{144}}{x_2^2} + \frac{A_{155}}{x_3^2} \right), \\ r_2 = \frac{A_{222}}{4d_2} \cdot \frac{1}{u_2} + \frac{u_2}{4d_2} \left(\frac{A_{233}}{x_1^2} + \frac{A_{244}}{x_2^2} \right), \\ r_3 = \frac{1}{2x_1} - \frac{1}{2d_3} \cdot \frac{1}{x_1^2} \left(u_0 \cdot A_{033} + u_1 \cdot A_{133} + u_2 \cdot A_{233} + x_2 \cdot A_{334} \right) + \frac{A_{345}}{2d_3} \left(\frac{x_1}{x_2x_3} - \frac{x_2}{x_1x_3} - \frac{x_3}{x_1x_2} \right), \\ r_4 = \frac{1}{2x_2} - \frac{1}{2d_4x_2^2} \left(u_0 \cdot A_{044} + u_1 \cdot A_{144} + u_2 \cdot A_{244} \right) + \frac{A_{334}}{4d_4} \left(\frac{x_2}{x_1^2} - \frac{2}{x_2} \right) + \frac{A_{345}}{2d_4} \left(\frac{x_2}{x_1x_3} - \frac{x_1}{x_2x_3} - \frac{x_3}{x_1x_2} \right), \\ r_5 = \frac{1}{2x_3} - \frac{1}{2d_5} \cdot \frac{1}{x_3^2} \left(u_0 \cdot A_{055} + u_1 \cdot A_{155} \right) + \frac{A_{345}}{2d_5} \left(\frac{x_3}{x_1x_2} - \frac{x_1}{x_3x_2} - \frac{x_2}{x_3x_1} \right). \end{array} \right.$$

The corresponding Ricci components of $F_4(2)$, $E_7(5)$ and $E_8(2)$ occur by the same expressions, by setting however $A_{144} = 0$.

5.2. The structure constants. We proceed now with the non-zero structure constants. We prove that

Lemma 5.3. *For the reductive decomposition (5.1) and for the left-invariant metric $\langle \cdot, \cdot \rangle$ on a Lie group $G = G(i_o)$ of Type $II_b(3)$ the non-zero triples A_{ijk} attain the following values:*

	A_{033}	A_{044}	A_{055}	A_{111}	A_{133}	A_{144}	A_{155}	A_{222}	A_{233}	A_{244}	A_{334}	A_{345}
$F_4(2)$	2/9	4/9	1/3	2/3	2	0	1/3	4/3	40/9	20/9	4	4/3
$E_7(5)$	5/18	5/9	1/6	1/3	5/2	0	1/6	35/3	140/9	70/9	10	5/3
$E_8(2)$	3/10	3/5	1/10	1/5	27/10	0	1/10	156/5	156/5	78/5	18	9/5
$E_7(3)$	2/9	4/9	1/3	20/3	12	4	4/3	4/3	40/9	20/9	10	10/3

Proof. We use the Killing metric to obtain a system of 5 equations $\{r_0 - r_1 = 0, r_1 - r_2 = 0, r_2 - r_3 = 0, r_3 - r_4 = 0, r_4 - r_5 = 0\}$ depending on 11 (or 12) unknowns, i.e. the triples appearing in Proposition 5.1. Two of them, namely the triples A_{334} and A_{345} can be computed similarly with a Lie group G of Type $I_b(3)$, using the (unique) Kähler-Einstein metric $x_1 = 1, x_2 = 2, x_3 = 3$ that $M = G/K$ admits. This gives (see [AnC])

$$(5.4) \quad A_{334} = \frac{d_3d_4 + 2d_3d_5 - d_4d_5}{d_3 + 4d_4 + 9d_5}, \quad A_{345} = \frac{(d_3 + d_4)d_5}{d_3 + 4d_4 + 9d_5}.$$

Case of $F_4(2), E_7(5), E_8(2)$. Assume that $G \in \{F_4(2), E_7(5), E_8(2)\}$. We will show how one can compute the other triples, i.e., $A_{033}, A_{044}, A_{055}, A_{111}, A_{133}, A_{144}, A_{155}, A_{222}, A_{233}$, and A_{244} , in a global way. For the construction of more equations, we use first the twistor fibration of our flag manifold over a symmetric space. Set

$$(5.5) \quad \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}, \quad \mathfrak{h} := \mathfrak{h}_1 \oplus \mathfrak{h}_2, \quad \mathfrak{h}_1 := \mathfrak{k}_0 \oplus \mathfrak{k}_2 \oplus \mathfrak{p}_2, \quad \mathfrak{h}_2 := \mathfrak{k}_1, \quad \mathfrak{n} := \mathfrak{p}_1 \oplus \mathfrak{p}_3.$$

This is a reductive decomposition of \mathfrak{g} with $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$, $[\mathfrak{h}, \mathfrak{n}] \subset \mathfrak{n}$ and $[\mathfrak{n}, \mathfrak{n}] \subset \mathfrak{h}$. Since $\mathfrak{k} \subset \mathfrak{h}$ we get a fibration $G/K \rightarrow G/H$, where $H \subset G$ is the connected Lie subgroup generated by \mathfrak{h} . The base space $B = G/H$ is an irreducible symmetric space and the fiber is a Hermitian symmetric space.

A second reductive decomposition of \mathfrak{g} is given by

$$(5.6) \quad \mathfrak{g} = \mathfrak{q} \oplus \mathfrak{r}, \quad \mathfrak{q} := \mathfrak{q}_1 \oplus \mathfrak{q}_2, \quad \mathfrak{q}_1 := \mathfrak{k}_0 \oplus \mathfrak{k}_1 \oplus \mathfrak{p}_3, \quad \mathfrak{q}_2 := \mathfrak{k}_2, \quad \mathfrak{r} := \mathfrak{p}_1 \oplus \mathfrak{p}_2.$$

In this case, the pair $(\mathfrak{g}, \mathfrak{q})$ is not symmetric, since $[\mathfrak{q}, \mathfrak{q}] \subset \mathfrak{q}$, $[\mathfrak{q}, \mathfrak{r}] \subset \mathfrak{r}$ but $[\mathfrak{r}, \mathfrak{r}] \subset (\mathfrak{q} \oplus \mathfrak{r}) = \mathfrak{g}$. It is $\mathfrak{r} = T_o' G/Q$ where $Q \subset G$ is the connected Lie subgroup with Lie algebra \mathfrak{q} . The fiber of the induced fibration is $\mathbb{C}P^2 \cong \mathrm{SU}_3/(\mathrm{U}_1 \times \mathrm{SU}_2) \cong \mathrm{SU}_3/\mathrm{U}_2$ and the base space $B' = G/Q$ is isotropy irreducible ([B]). Let us summarize the necessary details as follows:

The twistor fibration $G/K \rightarrow G/H$				The fibration $G/K \rightarrow G/Q$		
$G = G(i_o)$	\mathfrak{h}_1	\mathfrak{h}_2	$B = G/H$	\mathfrak{q}_1	\mathfrak{q}_2	$B' = G/Q$
$F_4(2)$	\mathfrak{sp}_3	\mathfrak{su}_2	$F_4/(\mathrm{Sp}_3 \times \mathrm{SU}_2)$	\mathfrak{su}_3	\mathfrak{su}_3	$F_4/(\mathrm{SU}_3 \times \mathrm{SU}_3)$
$E_7(5)$	\mathfrak{so}_{12}	\mathfrak{su}_2	$E_7/(\mathrm{SO}_{12} \times \mathrm{SU}_2)$	\mathfrak{su}_3	\mathfrak{su}_6	$E_7/(\mathrm{SU}_3 \times \mathrm{SU}_6)$
$E_8(3)$	\mathfrak{e}_7	\mathfrak{su}_2	$E_8/(\mathrm{E}_7 \times \mathrm{SU}_2)$	\mathfrak{su}_3	\mathfrak{e}_6	$E_8/(\mathrm{SU}_3 \times \mathrm{E}_6)$

The reductive decompositions (5.5) and (5.6) induce left-invariant metrics on G , given by

$$\langle\langle \cdot, \cdot \rangle\rangle = w_1 \cdot B|_{\mathfrak{h}_1} + w_2 \cdot B|_{\mathfrak{h}_2} + w_3 \cdot B|_{\mathfrak{n}}, \quad w_1, w_2, w_3 \in \mathbb{R}_+, \quad \langle\langle \cdot, \cdot \rangle\rangle' = z_1 \cdot B|_{\mathfrak{l}_1} + z_2 \cdot B|_{\mathfrak{l}_2} + z_3 \cdot B|_{\mathfrak{r}}, \quad z_1, z_2, z_3 \in \mathbb{R}_+,$$

respectively. For $w_1 = u_0 = u_2 = x_2$, $w_2 = u_1$ and $w_3 = x_1 = x_3$, the metrics $\langle\langle \cdot, \cdot \rangle\rangle$ and $\langle \cdot, \cdot \rangle$ coincide, and the same holds between $\langle\langle \cdot, \cdot \rangle\rangle'$ and $\langle \cdot, \cdot \rangle$ for $z_1 = u_0 = u_1 = x_3$, $z_2 = x_1$ and $z_3 = x_1 = x_2$. For these values, by comparing the Ricci components we get the relations $r_0 = r_2 = r_4$ and $r_0 = r_1 = r_5$. The first one $r_0 = r_2 = r_4$ and after introducing the values of A_{334}, A_{345} given by (5.4), implies that

$$\begin{aligned} A_{334} + 2A_{345} - d_4(A_{033} + A_{055}) &= d_2(A_{334} + 2A_{345}) - d_4A_{233}, \\ -A_{222} - A_{244} + d_2A_{044} &= 2(A_{044} + A_{244} + A_{334} + 2A_{345}) + d_4(A_{044} - 2), \\ 2(A_{044} + A_{244} + A_{334} + 2A_{345}) + d_4(A_{044} - 2) &= 2d_2(A_{044} + A_{244} + A_{334} + 2A_{345}) + d_4(A_{222} + A_{244} - 2d_2). \end{aligned}$$

Similarly, by $r_0 = r_1 = r_5$ we use the relations

$$\begin{aligned} A_{133} - d_1(A_{033} + A_{044}) &= -d_5(d_3(A_{133} - 2d_1) + d_4(4A_{133} - 2d_1) + 9d_5A_{133}), \\ A_{111} + A_{155} - A_{055} &= 2d_3(A_{055} + A_{155}) + 8d_4(A_{055} + A_{155}) + 18d_5(A_{055} + A_{155}) \\ &\quad + d_5(d_3A_{055} + 2d_3 + 4d_4(A_{055} - 1) + 9d_5(A_{055} - 2)). \end{aligned}$$

After combining now this data with the system defined by the Killing metric, we obtain all A_{ijk} in terms of the dimensions $d_i, i = 1, \dots, 5$ (recall that $d_0 = 1$); then one can complete the proof based on Table 3.

Case of $E_7(3)$. In this case, the first reductive decomposition is again the twistor fibration associated to the flag manifold $E_7/(\mathrm{U}_1 \times \mathrm{SU}_5 \times \mathrm{SU}_3)$. This is almost similar with (5.5), i.e. we set

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}, \quad \mathfrak{h} := \mathfrak{k} \oplus \mathfrak{p}_2 \cong \mathfrak{k} \oplus \mathfrak{m}_4, \quad \mathfrak{n} := \mathfrak{p}_1 \oplus \mathfrak{p}_3 \cong \mathfrak{m}_3 \oplus \mathfrak{m}_5.$$

It follows that $\mathfrak{h} \cong \mathfrak{su}_8 \subset \mathfrak{g}$ and since $[\mathfrak{n}, \mathfrak{n}] \subset \mathfrak{h}$ the base space of the fibration $G/K \rightarrow G/H$ is an irreducible symmetric space, namely $G/H \cong E_7/\mathrm{SU}_8$. For a second reductive decomposition we use (5.6); it is $\mathfrak{q} = \mathfrak{q}_1 \oplus \mathfrak{q}_2$ with $\mathfrak{q}_1 \cong \mathfrak{su}_6$ and $\mathfrak{q}_2 = \mathfrak{su}_3$. Thus we obtain the fibration

$$\mathbb{C}P^5 = \mathrm{SU}_6/\mathrm{U}_5 \rightarrow E_7/(\mathrm{U}_1 \times \mathrm{SU}_5 \times \mathrm{SU}_3) \rightarrow E_7/(\mathrm{SU}_6 \times \mathrm{SU}_3),$$

where the base space is isotropy irreducible ([B]). Considering new left-invariant metrics associated to these decompositions and following a similar procedure like before, we obtain the desired results. \square

Remark 5.4. Let us verify the values of A_{111}, A_{222} via Remark 4.5. For $F_4(2)$ recall that $B_{F_4}(\alpha_1, \alpha_1) = B_{F_4}(\alpha_2, \alpha_2) = 2B_{F_4}(\alpha_3, \alpha_3) = 2B_{F_4}(\alpha_4, \alpha_4)$. Because $\mathfrak{k}_1 = \mathfrak{su}_2$ is generated by α_1 , we see that $A_{111} = c \cdot \dim \mathfrak{su}_2$ where $c = B_{\mathrm{SU}_2}/B_{F_4} = 4/18$. Thus $A_{111} = 2/3$. For the triple A_{222} , the simple ideal $\mathfrak{k}_2 = \mathfrak{su}_3$ is generated by the short roots. Hence $A_{222} = \frac{1}{2}(c' \cdot \dim \mathfrak{su}_3)$ where $c' = B_{\mathrm{SU}_3}/B_{F_4} = 6/18$. This shows that $A_{222} = 4/3$. Similar are treated the other Lie groups. For example, for $E_7(3)$ we get $A_{111} = (B_{\mathrm{SU}_5}/B_{E_7}) \cdot \dim \mathfrak{su}_5 = (10/36) \cdot 24 = 2/3$ and $A_{222} = (B_{\mathrm{SU}_3}/B_{E_7}) \cdot \dim \mathfrak{su}_3 = (6/36) \cdot 8 = 4/3$. Although one is possible to use these values in the proof of Lemma 5.3, we remark that the reductive decompositions (5.5) and (5.6) described above, are both necessary.

5.3. Naturally reductive metrics. For a Lie group $G \cong G(i_o)$ of Type $II_b(3)$, left-invariant metrics on $G \cong G(i_o)$ which are $\text{Ad}(K)$ -invariant are given by

$$(5.7) \quad \langle \cdot, \cdot \rangle = u_0 \cdot B|_{\mathfrak{k}_0} + u_1 \cdot B|_{\mathfrak{k}_1} + u_2 \cdot B|_{\mathfrak{k}_2} + x_1 \cdot B|_{\mathfrak{p}_1} + x_2 \cdot B|_{\mathfrak{p}_2} + x_3 \cdot B|_{\mathfrak{p}_3}.$$

Proposition 5.5. *If a left invariant metric $\langle \cdot, \cdot \rangle$ of the form (5.7) on $G \cong G(i_o)$ of Type $II_b(3)$ is naturally reductive with respect to $G \times L$ for some closed subgroup L of G , then one of the following holds:*

(1) for $G = F_4(2)$, $E_7(5)$ and $E_8(2)$, $u_0 = u_2 = x_2$, $x_1 = x_3$, and for $G = E_7(3)$, $u_0 = u_1 = u_2 = x_2$, $x_1 = x_3$ (2) $u_0 = u_1 = x_3$, $x_1 = x_2$ (3) $x_1 = x_2 = x_3$.

Conversely, if one of the conditions (1), (2), (3) holds, then the metric $\langle \cdot, \cdot \rangle$ of the form (5.7) is naturally reductive with respect to $G \times L$, for some closed subgroup L of G .

Proof. Let \mathfrak{l} be the Lie algebra of L . Then there are two cases: $\mathfrak{l} \subset \mathfrak{k}$ or $\mathfrak{l} \not\subset \mathfrak{k}$. We begin with the second one, i.e. $\mathfrak{l} \not\subset \mathfrak{k}$. Let \mathfrak{h} be the subalgebra of \mathfrak{g} generated by \mathfrak{l} and \mathfrak{k} . Since $\mathfrak{g} = \mathfrak{k}_0 \oplus \mathfrak{k}_1 \oplus \mathfrak{k}_2 \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3$ is an irreducible decomposition as $\text{Ad}(K)$ -modules, the Lie algebra \mathfrak{h} needs to contain at least one of \mathfrak{p}_1 , \mathfrak{p}_2 , \mathfrak{p}_3 . Assume that $\mathfrak{p}_1 \subset \mathfrak{h}$. Then, $[\mathfrak{p}_1, \mathfrak{p}_1] \cap \mathfrak{p}_2 \neq \{0\}$ and hence \mathfrak{h} contains \mathfrak{p}_2 . It is also $[\mathfrak{p}_1, \mathfrak{p}_2] \cap \mathfrak{p}_3 \neq \{0\}$, thus \mathfrak{h} contains \mathfrak{p}_3 , as well. It follows that $\mathfrak{h} = \mathfrak{g}$ and the $\text{Ad}(L)$ -invariant metric $\langle \cdot, \cdot \rangle$ of the form (5.7) is bi-invariant. Now, if \mathfrak{h} contains \mathfrak{p}_2 , then $\mathfrak{h} \supset \mathfrak{k} \oplus \mathfrak{p}_2$. If $\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{p}_2$, then $(\mathfrak{h}, \mathfrak{p}_1 \oplus \mathfrak{p}_3)$ is a symmetric pair. Thus, the metric $\langle \cdot, \cdot \rangle$ of the form (5.7) satisfies $u_0 = u_2 = x_2$, $x_1 = x_3$ for $G = F_4(2)$, $E_7(5)$ and $E_8(2)$ and $u_0 = u_1 = u_2 = x_2$, $x_1 = x_3$ for $G = E_7(3)$. If $\mathfrak{h} \neq \mathfrak{k} \oplus \mathfrak{p}_2$, then it must be $\mathfrak{h} \cap \mathfrak{p}_1 \neq \{0\}$ or $\mathfrak{h} \cap \mathfrak{p}_3 \neq \{0\}$, so $\mathfrak{h} \supset \mathfrak{p}_1$ or $\mathfrak{h} \supset \mathfrak{p}_3$. Thus we also get $\mathfrak{h} = \mathfrak{g}$ and the $\text{Ad}(L)$ -invariant metric $\langle \cdot, \cdot \rangle$ of the form (5.7) is again bi-invariant. Consider now the case $\mathfrak{p}_3 \subset \mathfrak{h}$. Then, $\mathfrak{h} \supset \mathfrak{k} \oplus \mathfrak{p}_3$. If $\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{p}_3$, then \mathfrak{h} is a semi-simple Lie algebra and $\mathfrak{p}_1 \oplus \mathfrak{p}_2$ is an irreducible $\text{Ad}(H)$ -module. Thus, the metric $\langle \cdot, \cdot \rangle$ of the form (5.7) satisfies $u_0 = u_1 = x_3$, $x_1 = x_2$. If $\mathfrak{h} \neq \mathfrak{k} \oplus \mathfrak{p}_3$, we conclude that $\mathfrak{h} \cap \mathfrak{p}_1 \neq \{0\}$ or $\mathfrak{h} \cap \mathfrak{p}_2 \neq \{0\}$ and thus $\mathfrak{h} \supset \mathfrak{p}_1$, or $\mathfrak{h} \supset \mathfrak{p}_2$. Then, we obtain $\mathfrak{h} = \mathfrak{g}$ and the $\text{Ad}(L)$ -invariant metric $\langle \cdot, \cdot \rangle$ of the form (5.7) must be bi-invariant.

Now we consider the case $\mathfrak{l} \subset \mathfrak{k}$. Since the orthogonal complement \mathfrak{l}^\perp of \mathfrak{l} with respect to B contains the orthogonal complement \mathfrak{k}^\perp of \mathfrak{k} , it follows that $\mathfrak{l}^\perp \supset \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3$. Since the invariant metric $\langle \cdot, \cdot \rangle$ is naturally reductive with respect to $G \times L$, using Theorem 2.1. we conclude that $x_1 = x_2 = x_3$.

Conversely, if the condition (1) holds, then due to Theorem 2.1, the metric $\langle \cdot, \cdot \rangle$ given by (5.7) is naturally reductive with respect to $G \times L$, where $\mathfrak{l} = \mathfrak{k} \oplus \mathfrak{p}_2$. Similarly, if the condition (2) holds, then the metric given by (5.7) is naturally reductive with respect to $G \times L$, where $\mathfrak{l} = \mathfrak{k} \oplus \mathfrak{p}_3$. Finally, if the condition (3) holds, then the metric given by (5.7) is naturally reductive with respect to $G \times K$. \square

5.4. The homogeneous Einstein equation. Corollary 5.2 in combination with Lemma 5.3 determines now explicitly the Ricci tensor $\text{Ric}_{\langle \cdot, \cdot \rangle}$ of a Lie group $G = G(i_o)$ of Type $II_b(3)$ with respect to the left-invariant metric $\langle \cdot, \cdot \rangle$. Hence we can write down explicitly the homogeneous Einstein equation; this is given by

$$\{r_0 - r_1 = 0, \quad r_1 - r_2 = 0, \quad r_2 - r_3 = 0, \quad r_3 - r_4 = 0, \quad r_4 - r_5 = 0\}$$

and it turns out to be equivalent to the following system of equations (we normalise the metric by setting $x_1 = 1$).

Case of $F_4(2)$

$$(5.8) \quad \begin{cases} g_0 = 2u_0u_1x_2^2x_3^2 + 3u_0u_1x_2^2 + 4u_0u_1x_3^2 - 6u_1^2x_2^2x_3^2 - u_1^2x_2^2 - 2x_2^2x_3^2 = 0, \\ g_1 = 12u_1^2u_2x_2^2x_3^2 + 2u_1^2u_2x_2^2 - 10u_1u_2^2x_2^2x_3^2 - 5u_1u_2^2x_3^2 - 3u_1x_2^2x_3^2 + 4u_2x_2^2x_3^2 = 0, \\ g_2 = u_0u_2x_2^2x_3 + 9u_1u_2x_2^2x_3 + 50u_2^2x_2^2x_3 + 15u_2^2x_3 + 18u_2x_2^3x_3 + 6u_2x_2^3 - 108u_2x_2^2x_3 \\ \quad + 6u_2x_2x_3^2 - 6u_2x_2 + 9x_2^2x_3 = 0, \\ g_3 = -u_0x_2^2x_3 + 4u_0x_3 - 9u_1x_2^2x_3 - 20u_2x_2^2x_3 + 20u_2x_3 - 36x_2^3x_3 - 18x_2^3 + 108x_2^2x_3 \\ \quad + 6x_2x_3^2 - 72x_2x_3 + 18x_2 = 0, \\ g_4 = 9u_0x_2^2 - 4u_0x_3^2 + 9u_1x_2^2 - 20u_2x_3^2 + 18x_2^3x_3^2 + 48x_2^3x_3 - 108x_2^2x_3 - 48x_2x_3^3 \\ \quad + 72x_2x_3^2 + 24x_2x_3 = 0. \end{cases}$$

We consider the polynomial ring $R = \mathbb{Q}[z, u_0, u_1, u_2, x_2, x_3]$ and an ideal I , generated by polynomials $\{g_0, g_1, g_2, g_3, g_4, z u_0 u_1 u_2 x_2 x_3 - 1\}$. We take a lexicographic ordering $>$ with $z > u_0 > u_1 > u_2 > x_2 > x_3$ for a monomial ordering on R . Then, a Gröbner basis for the ideal I , contains a polynomial of x_3 given by

$(x_3 - 1)(884x_3^3 - 1816x_3^2 + 873x_3 - 117)h_1(x_3)$, where $h_1(x_3)$ is a polynomial of degree 101 defined by

$$\begin{aligned}
h_1(x_3) = & 63459125312728809061842327355883965092426940416x_3^{101} \\
& - 920983389392901147130377725860474388294846644224x_3^{100} \\
& + 9284030996847000229461748615161458859581745659904x_3^{99} \\
& - 72231264824781521074460759465312440396973648904192x_3^{98} \\
& + 474090092783696430917122803302705511377312635944960x_3^{97} \\
& - 2716426925303009081280894916587505354693138471452672x_3^{96} \\
& + 13928981666688388147256261223934160302674603191304192x_3^{95} \\
& - 64892996424133928137698638603811273070013563879292928x_3^{94} \\
& + 277731488039641471269947873533553174033919791213838336x_3^{93} \\
& - 1100681274916513923824472334739425330176822351407087616x_3^{92} \\
& + 4063212316482458197610601184936117632052406776092426240x_3^{91} \\
& - 14035457932119745594854722553431991322029237116190326784x_3^{90} \\
& + 45521154694910054415489734540334666446613275447418421248x_3^{89} \\
& - 138989344903939502760451623904236457989017703097989857280x_3^{88} \\
& + 400294158959755643083550785378988938431734533883781709824x_3^{87} \\
& - 1089022114669507770392870584714647593690007896934880641024x_3^{86} \\
& + 2801397856363975282367542926720029558144403125321202401280x_3^{85} \\
& - 6817885379922156536111652515165617977589266615874129231872x_3^{84} \\
& + 15701952412543110505856916943820752724521223983962586152960x_3^{83} \\
& - 34216393233291025770029652161256199220193387714745792659456x_3^{82} \\
& + 70516001329117409764548923930872952867194021910385077846016x_3^{81} \\
& - 137327646289169111738793490244260987895509460573594545225728x_3^{80} \\
& + 252415013953372334056500075226916999570602665659035875999744x_3^{79} \\
& - 437145843020198931924876541636921539338330946745999868755968x_3^{78} \\
& + 711698478115149919191326763960685722556799467931964286992384x_3^{77} \\
& - 1085854704561151226820526323366031976152058255866699795660800x_3^{76} \\
& + 1545872987530402035886865292388450940956043862142555058380800x_3^{75} \\
& - 2040770782009497991746315379021685834308361348137207206174720x_3^{74} \\
& + 2474617817759971607277241788391185316015028171144823683063808x_3^{73} \\
& - 2713416472372073086525101694784351591143139473401445118681088x_3^{72} \\
& + 2613262983470191771589830160198704851628875439784801973596160x_3^{71} \\
& - 2069290218418158264029847032239247119549407157373813654286336x_3^{70} \\
& + 1073308692563330553384597653734241243314674928772408635447296x_3^{69} \\
& + 243383268162901260317977997414016371083655623812595679834112x_3^{68} \\
& - 1609375404546416218236600482252912174118464079287975942948352x_3^{67} \\
& + 2672694963662822735196784395673190550399134349252116960776704x_3^{66} \\
& - 3114412610690042189775494926465007145115374286782216606602752x_3^{65} \\
& + 2780945528894563748863142343650355721268277028914024232564224x_3^{64} \\
& - 1777803153232519896953172291143481639817296527070092107145472x_3^{63} \\
& + 471376850935213838891432653962426924940360769359294745810432x_3^{62} \\
& + 623750383609419878697829019463867520660227080289808518313920x_3^{61} \\
& - 1040603038326989861022950362038120196827146648905065209736576x_3^{60} \\
& + 564965083609062720849342591878835952132818055988824262111376x_3^{59} \\
& + 649619372601281147814040345731771930965779727246024293143504x_3^{58} \\
& - 2118012753182384601180559819198400780972530222436309420548080x_3^{57} \\
& + 3205388926087602866149736984715502422941036840916282804140976x_3^{56} \\
& - 3399079409763580385139579223928130084953706063347065837023000x_3^{55} \\
& + 2526218612878420387036879439950686812518557153001711837979064x_3^{54} \\
& - 834433401752454857645647262023568632488618740772182979146772x_3^{53} \\
& - 1101385395838477492629746207491960938518647539595102389316872x_3^{52} \\
& + 2631037139614283301382941342160092654160835574746224378985565x_3^{51} \\
& - 3282383226236719186391399204082397795775877578301500715196503x_3^{50} \\
& + 2937659835203687649893255383322025931399696823727244558286097x_3^{49} \\
& - 1851035734616115365629019524331368234044251593409402913444411x_3^{48} \\
& + 498022109054373150395850625480317180124853810659501884348651x_3^{47} \\
& + 627355526093354022331902293368985581146006710909004658401207x_3^{46} \\
& - 1207045734980737886266791122600422887565869937071216149025129x_3^{45} \\
& + 1182073307241943169778530232935546805223288647965360710571511x_3^{44} \\
& - 718203246621926363352199146463822746343025839992293567678988x_3^{43} \\
& + 96644002741648960970123374601314631036176300051445193949656x_3^{42} \\
& + 422104069724220785760650059880175853561682549627138288223924x_3^{41} \\
& - 683397637477216946107242310612442756271548207816068470401088x_3^{40}
\end{aligned}$$

$$\begin{aligned}
& +666605501872783619736009469651934045513775273586944371856186x_3^{39} \\
& -452558710835995517519348198954994152927678719275198879680886x_3^{38} \\
& +162700999764271444384776912405378431410116935139824677075998x_3^{37} \\
& +93939119293706672636919053860282946725625620077832878556914x_3^{36} \\
& -255147629760439396399382896402394067054928161056011639874616x_3^{35} \\
& +308396407398907558760838372721158434444550638676652132806780x_3^{34} \\
& -277614375770280461907766589637738690110973121068605430180588x_3^{33} \\
& +201477143418161317806510553487993326097416754880917183892864x_3^{32} \\
& -115969901716816418101426110321015198051414624540044967024168x_3^{31} \\
& +44725833423519690527789412859895224361666978693724364777728x_3^{30} \\
& +2520894187976416901089600003713269486821645786597713677964x_3^{29} \\
& -26486403386554915622730996860937772397953223304069070744920x_3^{28} \\
& +33294891175701376903563757255573808063712434139603578954691x_3^{27} \\
& -3020207688496188736282779340403368100144991071060222187421x_3^{26} \\
& +23099423829744618419506899206593906790694209676039929614503x_3^{25} \\
& -15680468836461732445911867905683122953673941068247037316197x_3^{24} \\
& +9671984604335042749855869480653845526481625108821701237745x_3^{23} \\
& -5490069835615675038150115079186954852016192157885614205023x_3^{22} \\
& +2889374986224761685442552566105705408525169491706393088757x_3^{21} \\
& -1416494130573647866724791573304034482931984659037915662955x_3^{20} \\
& +648751477566144492286603717146642412127675425476232549029x_3^{19} \\
& -278070848833909095692181583970081277475757069716551459039x_3^{18} \\
& +111644780007072437002497603341623817366814844891773288945x_3^{17} \\
& -41997637999280362408255038869439084693844361764781722551x_3^{16} \\
& +14796365048687965062327228402196888338897351487134760371x_3^{15} \\
& -4877819943619247140179803803393490676071239723633511361x_3^{14} \\
& +1502397252705966074311686966159213579712203646293106491x_3^{13} \\
& -431425769917898886316566505773997593134379146416278993x_3^{12} \\
& +115173309907631982466727249550402897704325044395195191x_3^{11} \\
& -28477674526413418871108871989347987750566901612750701x_3^{10} \\
& +6490427377397192429051997305835939685976063245759899x_3^9 \\
& -1355038458126617854787910001663192486402952805121845x_3^8 \\
& +257038525794588371891326953346269064271241145105863x_3^7 \\
& -43821959983623391384006935940560441466837308645513x_3^6 \\
& +661571885884967733806801498204423491247659359667x_3^5 \\
& -865800811288635528014727851709253657063913408253x_3^4 \\
& +95113645509127999488018808217880353172800724072x_3^3 \\
& -8317620756384461283345771094408286533995673140x_3^2 \\
& +522260976842192193467930459675557760528309268x_3 \\
& -17941225061011393318805831702298275346720564.
\end{aligned}$$

Solving $h_1(x_3) = 0$ numerically, we see that there exist five positive solutions, which are given approximately by $x_3 \approx 0.25594917$, $x_3 \approx 0.49280351$, $x_3 \approx 1.1060677$, $x_3 \approx 1.3849054$, $x_3 \approx 2.4753269$. Moreover, real solutions of the system $\{g_0 = 0, g_1 = 0, g_2 = 0, g_3 = 0, g_4 = 0, h_1(x_3) = 0\}$ with $u_0 u_1 u_2 x_2 x_3 \neq 0$ are of the form

$$\begin{aligned}
& \{u_0 \approx 0.26967359, u_1 \approx 0.21126932, u_2 \approx 0.11447898, x_2 \approx 1.0039269, x_3 \approx 0.25594917\}, \\
& \{u_0 \approx 0.54675598, u_1 \approx 0.28246829, u_2 \approx 1.1715986, x_2 \approx 1.022763119985955, x_3 \approx 0.49280351\}, \\
& \{u_0 \approx 0.72660638, u_1 \approx 0.143286728, u_2 \approx 0.11340863, x_2 \approx 0.49841277, x_3 \approx 1.1060677\}, \\
& \{u_0 \approx 1.3613346, u_1 \approx 1.4496103, u_2 \approx 0.15582596, x_2 \approx 0.95415861, x_3 \approx 1.3849054\}, \\
& \{u_0 \approx 2.4948349, u_1 \approx 0.25221774, u_2 \approx 0.17749081, x_2 \approx 1.7461504, x_3 \approx 2.4753269\}.
\end{aligned}$$

Hence, we obtain five Einstein metrics which are non-naturally reductive by Proposition 5.5. By computing the associated scale invariants (cf. [AC]) we conclude that these five metrics are non-isometric each other.

For $884x_3^3 - 1816x_3^2 + 873x_3 - 117 = 0$, we see that $x_2 = 1$, $u_0 = u_1 = x_3$ and $442x_3^2 - 739x_3 + 65u_2 + 144 = 0$ are solutions of the system $\{g_0 = 0, g_1 = 0, g_2 = 0, g_3 = 0, g_4 = 0\}$. Hence, in this case solutions are given by

$$\begin{aligned}
& \{u_0 = u_1 = x_3 \approx 0.23910517, u_2 \approx 0.11429253, x_2 = 1\}, \\
& \{u_0 = u_1 = x_3 \approx 0.38779156, u_2 \approx 1.1709075, x_2 = 1\}, \\
& \{u_0 = u_1 = x_3 \approx 1.4274019, u_2 \approx 0.15823885, x_2 = 1\}.
\end{aligned}$$

For $x_3 = 1$, we see that $(x_2 - 1)(2375x_2^3 - 4195x_2^2 + 1960x_2 - 272) = 0$, $u_0 = u_2 = x_2$ and $102u_1 - 2375x_2^3 + 6570x_2^2 - 5305x_2 + 1008 = 0$. Then we obtain again solutions, approximately given by

$$\begin{aligned} \{u_0 = u_2 = x_2 \approx 0.27971768, u_1 \approx 0.13559746, x_3 = 1\}, \\ \{u_0 = u_2 = x_2 \approx 0.36506883, u_1 \approx 1.6532011, x_3 = 1\}, \\ \{u_0 = u_2 = x_2 \approx 1.1215293, u_1 \approx 0.27621689, x_3 = 1\} \end{aligned}$$

and $u_0 = u_1 = x_2 = x_3 = 1$. Notice that these solutions define naturally reductive Einstein metrics, by Proposition 5.5.

Case of $E_7(5)$

$$(5.9) \quad \left\{ \begin{aligned} g_0 &= 5u_0u_1x_2^2x_3^2 + 3u_0u_1x_2^2 + 10u_0u_1x_3^2 - 15u_1^2x_2^2x_3^2 - u_1^2x_2^2 - 2x_2^2x_3^2 = 0, \\ g_1 &= 15u_1^2u_2x_2^2x_3^2 + u_1^2u_2x_2^2 - 8u_1u_2^2x_2^2x_3^2 - 4u_1u_2^2x_3^2 - 6u_1x_2^2x_3^2 + 2u_2x_2^2x_3^2 = 0, \\ g_2 &= u_0u_2x_2^2x_3 + 9u_1u_2x_2^2x_3 + 104u_2^2x_2^2x_3 + 24u_2^2x_3 + 36u_2x_2^3x_3 + 6u_2x_2^3 \\ &\quad - 216u_2x_2^2x_3 + 6u_2x_2x_3^2 - 6u_2x_2 + 36x_2^2x_3 = 0, \\ g_3 &= -u_0x_2^2x_3 + 4u_0x_3 - 9u_1x_2^2x_3 - 56u_2x_2^2x_3 + 56u_2x_3 - 72x_2^3x_3 - 18x_2^3 \\ &\quad + 216x_2^2x_3 + 6x_2x_3^2 - 144x_2x_3 + 18x_2 = 0, \\ g_4 &= 9u_0x_2^2 - 4u_0x_3^2 + 9u_1x_2^2 - 56u_2x_3^2 + 36x_2^3x_3^2 + 102x_2^3x_3 \\ &\quad - 216x_2^2x_3 - 102x_2x_3^3 + 144x_2x_3^2 + 78x_2x_3 = 0. \end{aligned} \right.$$

We consider the polynomial ring $R = \mathbb{Q}[z, u_0, u_1, u_2, x_2, x_3]$ and an ideal I generated by polynomials $\{g_0, g_1, g_2, g_3, g_4, z u_0 u_1 u_2 x_2 x_3 - 1\}$. Fix a lexicographic ordering $>$, with $z > u_0 > u_1 > u_2 > x_2 > x_3$ for a monomial ordering on R . Then, a Gröbner basis for the ideal I contains a polynomial of x_3 given by $(x_3 - 1)(2332x_3^3 - 4013x_3^2 + 1053x_3 - 72)h_1(x_3)$, where $h_1(x_3)$ is a polynomial of degree 101 given by

$$\begin{aligned} h_1(x_3) = & 140762503345409851806498624344165883958339691162109375000000000000000x_3^{101} \\ & - 1718791847963894825198460893177825091406702995300292968750000000000000x_3^{100} \\ & + 145313195810420524500555185389570281512774527072906494140625000000000000x_3^{99} \\ & - 947333138737497893055713559238267540364741161465644836425781250000000000x_3^{98} \\ & + 522284290327260089043569633944770817557391159236431121826171875000000000x_3^{97} \\ & - 2515439510033906451541094833423024544136197418138384819030761718750000000x_3^{96} \\ & + 10856130009524847880321821565672055732647923935993313789367675781250000000x_3^{95} \\ & - 42588824593596186405769169998591982296086941388526707887649536132812500000x_3^{94} \\ & + 153533817922010931692614427457285999744692771022045439481735229492187500000x_3^{93} \\ & - 5124767143797262421903507509416017458235062360763549804687500000x_3^{92} \\ & + 159277968193531827733770928938696792394519161440710672831535339355468750000x_3^{91} \\ & - 462926508346144498742728870248893592374879475272611933228969573974609375000x_3^{90} \\ & + 1262177999422453138852307438542980370000112170664719322771072387695312500000x_3^{89} \\ & - 3236435366700326861977880348547854838802610550418218812595033645629882812500x_3^{88} \\ & + 7818839674582226915745074206106944841667485955194252036326932907104492187500x_3^{87} \\ & - 17823026475683574989920756727849914992828758644347139452949285984039306640625x_3^{86} \\ & + 38374562464573820491352405963633381848692969526613948014654363346099853515625x_3^{85} \\ & - 78111515156680952208568322815283640971150078626459790696200255393981933593750x_3^{84} \\ & + 15041821286624939274643013613647719031218213134670864178384834232330322656250x_3^{83} \\ & - 27422679779220136068829653849816447018158152571233876100911887187957763671875x_3^{82} \\ & + 473666436371352173985976176338257382301103274684463812827026091655731201171875x_3^{81} \\ & - 775937848157366756906063084448422279134097874371188256677871034072875976562500x_3^{80} \\ & + 1207150400502827505609484743337103792488617773663813184312931088278198242187500x_3^{79} \\ & - 1786917539761664393734512028432217575971928747001709362696207536544036865234375x_3^{78} \\ & + 2523283328042500705555771937323177208536480253814622741933239530885467529296875x_3^{77} \\ & - 3410040133101191376307418153187605141034005485155297795948203643118286132812500x_3^{76} \\ & + 4427227829007830039323985782362477540309186880924655617991484124562622070312500x_3^{75} \\ & - 5543315902885880050459899218485938422135193995167025320528375071687469482421875x_3^{74} \\ & + 6716145578402019256272307762678559235656688619049543258249088075475286865234375x_3^{73} \end{aligned}$$

$-788932541552162231631911701476295930690766787110087001641089683142939453125000x_3$ ⁷²
 $+8987871554017934014648949314880532895050225239219525645069037648161762695312500x_3$ ⁷¹
 $-9917208157477374337767877875545519619987011928976120543819610153102255859375000x_3$ ⁷⁰
 $+10577799203698319620387455592337741027001238708759415658121587619127446289062500x_3$ ⁶⁹
 $-10887319782215050901949644489831112625025397307880553428016972363473247070312500x_3$ ⁶⁸
 $+10808272718671764880409307324632946987250904801571361749421569584932820507812500x_3$ ⁶⁷
 $-10351925750612475241625493181878415851036122298454696642632323147793814306640625x_3$ ⁶⁶
 $+9570049235629322030129441573282410555362644512739858159923675452999522548828125x_3$ ⁶⁵
 $-8527728410757032303006877739950121077903047743081938287293482897667003828125000x_3$ ⁶⁴
 $+7293671555087344683446136873087134513658482869052893430821710819055271882812500x_3$ ⁶³
 $-5938364201902229269149097008903216497897458637908470944612521462398610144531250x_3$ ⁶²
 $+4542803163100520326399144842143300663460846346791886417852196110264118186718750x_3$ ⁶¹
 $-3203367797363298249711802033233747850433277119218375865387587239202504776562500x_3$ ⁶⁰
 $+2014757550707134984844606694758564217815440179528736200907390078758647693437500x_3$ ⁵⁹
 $-1059435189706540868544396172374688428356034633803064566575279632528066663046875x_3$ ⁵⁸
 $+373453746953392113493477521723258123175987786047445920972626980195467362484375x_3$ ⁵⁷
 $+44530362210518073353216396492748240313013722867507760033999699035992575625000x_3$ ⁵⁶
 $-245551311625611100016210619803011492135965095443877459657793044352671972512500x_3$ ⁵⁵
 $+289708058711370824511848194485259992761349222486398117737364263666816009706250x_3$ ⁵⁴
 $-251021510422094576034704205162129823203080952220679044063099876553064746298750x_3$ ⁵³
 $+176974180980720883702202869219411157489456320399570126363647370079444394210000x_3$ ⁵²
 $-107293425749110552425481777607895918800703150070799670761477099264571385035000x_3$ ⁵¹
 $+52661105129970562594028210108639606728701622938738221071821689744328675071625x_3$ ⁵⁰
 $-20209976916043807751745039654358855823532177997588147118317671488142414778125x_3$ ⁴⁹
 $+2385419029486295607532750803216893332123501222180408286275540947211555452000x_3$ ⁴⁸
 $+3587821317546678618380217860464824345029642368821605269921381443889756146300x_3$ ⁴⁷
 $-5067035401021094823238464500969069646022963960363015867883320197337727649300x_3$ ⁴⁶
 $+3717083826554658487879634379063790156238599455962131271814245768935578843000x_3$ ⁴⁵
 $-2427908795670998175497265078142249597119437614024384136832425990803168150860x_3$ ⁴⁴
 $+1195740741369749034292959741455193545682210186321696447733419655241996493300x_3$ ⁴³
 $-547903002336498666464976405918498018000843427855146404566963006900577627145x_3$ ⁴²
 $+168574323027718935579747256538556505399257352625388808341195401828470961037x_3$ ⁴¹
 $-3165649270926734393346707272620048507992779716620141360266989747001709088x_3$ ⁴⁰
 $-19141463779266076060464696963922265617567912033126227666771741940037097492x_3$ ³⁹
 $+22057805196225616222038220769784003421664886692244997020731817495990602040x_3$ ³⁸
 $-16111103356817483342224330398745646301705527199879085417642376254012031316x_3$ ³⁷
 $+8787371168033807690276772563251134119884622902813540153994393586109768596x_3$ ³⁶
 $-4203136839615520864011534300079602987537250849136320058974663974205829364x_3$ ³⁵
 $+1703272099354575963895402631669026840927858600614506299961357771268340029x_3$ ³⁴
 $-588970874392788182709636324404656170014408669935999543313372542768494777x_3$ ³³
 $+153806609923832326823604769432015034921034937840431892763207363982129000x_3$ ³²
 $-15378470715780405849560366161840779036747529936249773865278808437342372x_3$ ³¹
 $-15211919485197391079635722664817141582826291606688169501432074860152694x_3$ ³⁰
 $+14602884726425854387595532983100243969072684905051532652335375162018618x_3$ ²⁹
 $-8742436435221198517785178525746417053733167289492701877870544777319580x_3$ ²⁸
 $+4322817710270412971587127477714461997494242365934113140183874504735076x_3$ ²⁷
 $-1894810327329607561294394972277312203013169914051239516934551367169545x_3$ ²⁶
 $+759207200812335120180804124931553087396041536632050200202305980399381x_3$ ²⁵
 $-282425511639204787872546053183174770397888747384260698976130175319380x_3$ ²⁴
 $+98377870025599081710637608493372281528211731859183399640998528484952x_3$ ²³
 $-32273084568675355240409087819690258656183993127476521438269949061881x_3$ ²²
 $+1000189548315273268401949436959539522575021799838928653085888305093x_3$ ²¹
 $-2934440317716499911008283722824705734798918191533548524562464809626x_3$ ²⁰
 $+815886497078810400694413343083008136001965170879611485375752520798x_3$ ¹⁹
 $-215013629240317734592312901320891652601683163107174766915835738002x_3$ ¹⁸
 $+53694155253757070364771839375415493996163225978928643803111746886x_3$ ¹⁷
 $-12693754793801484385740806712618680578641191192421808450522707324x_3$ ¹⁶
 $+2836871266322051379178111812447045352149407143579909976950226088x_3$ ¹⁵
 $-598220909132580616232973267331034636299134258811161962399579373x_3$ ¹⁴
 $+118729056990130109140794450645759854158605184235508731658294357x_3$ ¹³
 $-22112260233460135927357201183413456647262866287675952659637328x_3$ ¹²
 $+3850068285596588964841282201703973992205738072480521413341576x_3$ ¹¹
 $-623837287469755356862826730619025821947727433644178976756360x_3$ ¹⁰

$$\begin{aligned}
& +9354843836484139280769398239228207317040392100878886330336x_3^9 \\
& -12891115269601460910598971287468238872883412219299739419408x_3^8 \\
& +1617872450840116983877791561306647646149713087425933514704x_3^7 \\
& -182775085352679354205708790407982060766987938597053694592x_3^6 \\
& +18291418108089247071802150845244940952239814288051333120x_3^5 \\
& -1584767080810168250170572570688441271916400990819270656x_3^4 \\
& +114880970013053787307549156392012937701059689924067328x_3^3 \\
& -6569798492576356809368814625199214133359843746512896x_3^2 \\
& +263729603959951432986035759935455148324708203626496x_3 \\
& -5546140081463900301505979788442876096704520650752
\end{aligned}$$

Solving $h_1(x_3) = 0$ numerically, we get five positive and four negative solutions, given approximately by $x_3 \approx 1.1800573$, $x_3 \approx 0.12169301$, $x_3 \approx 0.20754861$, $x_3 \approx 1.5303652$, $x_3 \approx 2.1692738$ and $x_3 \approx -0.49217418$, $x_3 \approx -0.48405841$, $x_3 \approx -0.57150512$, $x_3 \approx -0.6231710$. In addition, we see that real solutions of the system $\{g_0 = 0, g_1 = 0, g_2 = 0, g_3 = 0, g_4 = 0, h_1(x_3) = 0\}$ with $u_0 u_1 u_2 x_2 x_3 \neq 0$, are given by

$$\begin{aligned}
& \{u_0 \approx 0.96224102, u_1 \approx 0.073621731, u_2 \approx 0.24880488, x_3 \approx 1.1800573, x_2 \approx 0.66994762\}, \\
& \{u_0 \approx 0.12590407, u_1 \approx 0.10843941, u_2 \approx 0.24541305, x_3 \approx 0.12169301, x_2 \approx 1.0005214\}, \\
& \{u_0 \approx 0.22329332, u_1 \approx 0.15361011, u_2 \approx 1.1356290, x_3 \approx 0.20754861, x_2 \approx 1.0023024\}, \\
& \{u_0 \approx 1.5711912, u_1 \approx 1.3666215, u_2 \approx 0.31577380, x_3 \approx 1.5303652, x_2 \approx 1.1132523\}, \\
& \{u_0 \approx 2.1869813, u_1 \approx 0.10309808, u_2 \approx 0.33851600, x_3 \approx 2.1692738, x_2 \approx 1.5911459\}
\end{aligned}$$

and

$$\begin{aligned}
& \{u_0 \approx 0.77867456, u_1 \approx 0.18058466, u_2 \approx 1.0076703, x_3 \approx -0.49217418, x_2 \approx 2.8011127\}, \\
& \{u_0 \approx 0.75556174, u_1 \approx 0.56064249, u_2 \approx 0.94565380, x_3 \approx -0.48405841, x_2 \approx 2.8687188\}, \\
& \{u_0 \approx 1.0814568, u_1 \approx 0.76564465, u_2 \approx 0.48544174, x_3 \approx -0.57150512, x_2 \approx 3.1347988\}, \\
& \{u_0 \approx 1.2615164, u_1 \approx 0.13352863, u_2 \approx 0.44039770, x_3 \approx -0.62317102, x_2 \approx 3.1494041\}.
\end{aligned}$$

Thus, we obtain five Einstein metrics which are non-naturally reductive by Proposition 5.5. We can also see that these five metrics are non-isometric each other, by computing the induced scale invariants (cf. [AC]).

For $2332x_3^3 - 4013x_3^2 + 1053x_3 - 72 = 0$, we see that $x_2 = 1$, $u_0 = u_1 = x_3$ and $2332x_3^3 - 6345x_3^2 + 4642x_3 - 224u_2 - 405 = 0$ are solutions of the system of equations $\{g_0 = 0, g_1 = 0, g_2 = 0, g_3 = 0, g_4 = 0\}$. Thus, approximately we obtain the following solutions of the system corresponding to the homogeneous Einstein equation:

$$\begin{aligned}
& \{u_0 = u_1 = x_3 \approx 0.11699044, u_2 \approx 0.24536236, x_2 = 1\}, \\
& \{u_0 = u_1 = x_3 \approx 0.18615305, u_2 \approx 1.1352348, x_2 = 1\}, \\
& \{u_0 = u_1 = x_3 \approx 1.4176970, u_2 \approx 0.30406198, x_2 = 1\}.
\end{aligned}$$

For $x_3 = 1$, we see that $(x_2 - 1)(4949x_2^3 - 9379x_2^2 + 5155x_2 - 875) = 0$, $u_0 = u_2 = x_2$ and $525u_1 - 19796x_2^3 + 57312x_2^2 - 49561x_2 + 11520 = 0$. In this case, we obtain the solutions

$$\begin{aligned}
& \{u_0 = u_2 = x_2 \approx 0.37412457, u_1 \approx 0.069921978, x_3 = 1\}, \\
& \{u_0 = u_2 = x_2 \approx 0.43525576, u_1 \approx 1.5741577, x_3 = 1\}, \\
& \{u_0 = u_2 = x_2 \approx 1.0857500, u_1 \approx 0.12593450, x_3 = 1\}
\end{aligned}$$

and $u_0 = u_1 = x_2 = x_3 = 1$. By Proposition 5.5, we see that these solutions induce left-invariant Einstein metrics which are naturally reductive.

Case of $E_8(2)$

(5.10)

$$\left\{ \begin{aligned}
g_0 &= 9u_0u_1x_2^2x_3^2 + 3u_0u_1x_2^2 + 18u_0u_1x_3^2 - 27u_1^2x_2^2x_3^2 - u_1^2x_2^2 - 2x_2^2x_3^2 = 0, \\
g_1 &= 27u_1^2u_2x_2^2x_3^2 + u_1^2u_2x_2^2 - 12u_1u_2^2x_2^2x_3^2 - 6u_1u_2^2x_3^2 - 12u_1x_2^2x_3^2 + 2u_2x_2^2x_3^2 = 0, \\
g_2 &= u_0u_2x_2^2x_3 + 9u_1u_2x_2^2x_3 + 176u_2^2x_2^2x_3 + 36u_2^2x_3 + 60u_2x_2^3x_3 + 6u_2x_2^3 \\
&\quad - 360u_2x_2^2x_3 + 6u_2x_2x_3^2 - 6u_2x_2 + 72x_2^2x_3 = 0, \\
g_3 &= -u_0x_2^2x_3 + 4u_0x_3 - 9u_1x_2^2x_3 - 104u_2x_2^2x_3 + 104u_2x_3 - 120x_2^3x_3 - 18x_2^3 \\
&\quad + 360x_2^2x_3 + 6x_2x_3^2 - 240x_2x_3 + 18x_2 = 0, \\
g_4 &= 9u_0x_2^2 - 4u_0x_3^2 + 9u_1x_2^2 - 104u_2x_2^2 + 60x_2^3x_3^2 + 174x_2^3x_3 - 360x_2^2x_3 \\
&\quad - 174x_2x_3^3 + 240x_2x_3^2 + 150x_2x_3 = 0.
\end{aligned} \right.$$

Consider the polynomial ring $R = \mathbb{Q}[z, u_0, u_1, u_2, x_2, x_3]$ and an ideal I , generated by polynomials $\{g_0, g_1, g_2, g_3, g_4, z u_0 u_1 u_2 x_2 x_3 - 1\}$. Fix the lexicographic ordering $>$, with $z > u_0 > u_1 > u_2 > x_2 > x_3$ for a monomial ordering on R . Then, by the aid of computer, we compute a Gröbner basis for the ideal I ; this contains a polynomial of x_3 given by $(x_3 - 1)(14863x_3^3 - 23537x_3^2 + 3841x_3 - 159)h_1(x_3)$, where $h_1(x_3)$ is a polynomial of degree 101 given by

$$\begin{aligned}
h_1(x_3) = & 4300817068102634554892991062842129042773576732311618329635448320000000000x_3^{101} \\
& - 48434234613549258526882893300617941462391685037263593180733084672000000000x_3^{100} \\
& + 3844124140008351534979627323358823059356059184250272275353817688499200000000x_3^{99} \\
& - 2343356739326681041821514580356528768679045141260294555070279790135552000000x_3^{98} \\
& + 1211875039247632622236533006962244871827435009196838279324500368662097280000x_3^{97} \\
& - 54694072490084700582663695979794196000771759437095459232706976031069894016000x_3^{96} \\
& + 221353224406881636197442133058784492973200290115121516088278948796937464371200x_3^{95} \\
& - 813886740762282999039919900837623187565903920555774696522128146040136921808640x_3^{94} \\
& + 2749311143142358208694757864933801593004701230356043145755440407202107969205248x_3^{93} \\
& - 8592602645047256285489822902989371514808504859611203756617492188201050618429696x_3^{92} \\
& + 24987213573707362099498581598734764124216513406921443771718443643063519416173424x_3^{91} \\
& - 67886371151784002113561327332159978566678061704085234326096771582310691912274832x_3^{90} \\
& + 172855472915888793901915966837849077301067956523296460430225355574530069733610000x_3^{89} \\
& - 413520597399357357100869381565342441018263038178115494696023037473035005867485760x_3^{88} \\
& + 931201134042478123783110021283671383867077991181025700457605569636368192762700536x_3^{87} \\
& - 1977180680461234875709924061155165455728674465172548132743172377066740066067927072x_3^{86} \\
& + 3963718711108383180414158119936397189175742439421867408680408165110894988557985004x_3^{85} \\
& - 7513418122634704515990854605187661012865820116011720424270638810049480025748301056x_3^{84} \\
& + 13485034115259604979202660060065426293023734909520325939447188838010611615994227899x_3^{83} \\
& - 22953711751263858356155615317993870971118178202495574682769731600848840969284513211x_3^{82} \\
& + 37118739850644510331943371356546784531091762451361904566881906562825838499125532570x_3^{81} \\
& - 57143381922307096358894776517481035852112895921488424717999575316225143862664779938x_3^{80} \\
& + 83933828799625827613068937568371537392889738211567188929892877463435790674461594289x_3^{79} \\
& - 1179168491898124486573523620100796183753738926756173481970307068184674344590652785729x_3^{78} \\
& + 158840601295945663967771914515932889808801362497376498082318501776572544579687584532x_3^{77} \\
& - 205640403374951186570968875180984604118953468621671855171500403944532124984014864228x_3^{76} \\
& + 256373362974138627914650186250703319027584745315995489017313493311167581918499221209x_3^{75} \\
& - 308180457016417463546322721674440420609583883456596081505287450408132053995421688417x_3^{74} \\
& + 357437554714305319709984089082160479773294679519620215685747593181661184213216972570x_3^{73} \\
& - 399909639063653725636419295835615541787351965342966672935569109836991011645113298530x_3^{72} \\
& + 431447528558011049848674754343083702264157712308167972064641240229606934035503112751x_3^{71} \\
& - 448435605178939066244729947066284792252456509588375968250745873651107135576456816847x_3^{70} \\
& + 448909425142372532200712964875932843329902983017003297057000758414145225424316299356x_3^{69} \\
& - 432505986718923727726733642983121597245375455185536827443718365457823110316105606920x_3^{68} \\
& + 400926851409634138076302850835949788171629035117892773365279927750703254819903773328x_3^{67} \\
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& + 304117822268847097534981499298924198481044422812969119854167753513347346281757734984x_3^{65} \\
& - 246800083651937571501758366885734742867433138789154590922218902161270727737822408680x_3^{64} \\
& + 189342447332993406197869816861229081733498325675444469649008978221731371770480267524x_3^{63} \\
& - 136571233453352364375287398041234776929197601714558372362384908184149357492228993124x_3^{62} \\
& + 9161417425254103618326117195316985148914524584637790106371818724276021551614226124x_3^{61} \\
& - 57099174743574573249126209346809266661445782155473938421471474074519986140555333552x_3^{60} \\
& + 32358254824131527171676058875969221067276651159847698655117241266482730946477408340x_3^{59} \\
& - 16824644814321478644964952093517021134010663065144141852879740223864734400865575220x_3^{58} \\
& + 7514287187418446688509325329918072893001195320661832719856558857169013514235203720x_3^{57} \\
& - 2984410089768692604131117349590773663180032424321033953036482267179033266809725640x_3^{56} \\
& + 732445676639314254349675396500675892268684514864442230285668149583939903578943676x_3^{55} \\
& - 67917208841309119819997191749712546424834785848702592652838825458694211721954412x_3^{54} \\
& - 201226807780484731518790106748244287589322324607269243036806074712412930566274184x_3^{53} \\
& + 129154056812060487385840015218861595472761603600998499969817431810471728718352640x_3^{52} \\
& - 107834101343192331330480731011834920513318003452547815351643422543284825020233542x_3^{51} \\
& + 39139302564982238988531563687231140402316126213769637000710966449760382422072550x_3^{50} \\
& - 25014007394124148943263880664666605315368407336198528031642273872797438993869524x_3^{49} \\
& + 5149637394602359328977892265587734288759270529004199895668361076640377995064036x_3^{48} \\
& - 3349820696141993438064008629889341109854136879528713117453652408803499479096386x_3^{47} \\
& - 85257700269450629092725979032910336876774470641715409238975393462391067569870x_3^{46}
\end{aligned}$$

$$\begin{aligned}
& -172220176540080050421320403630494784694130728150815944148121248793493036348296x_3^{45} \\
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& +29933044025820518391774124363260016171392605814344766308091877183115582225214x_3^{43} \\
& -4072144967747316245823018558883004422340055565697639872956766127283187956142x_3^{42} \\
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& -5189717122288769398584568108177516806684072008723652373050127514128137067260x_3^{40} \\
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& -369906466718587873259473075984672842769304229881787413153149704347082720882x_3^{38} \\
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& -8953116075124778902766759252670482762033989180259875789742764174595473548x_3^{35} \\
& +3507574455024951317268532533764032607025789087442517730121154962348230396x_3^{34} \\
& -1506654748146747600991905293586765340051270768797114324562291232201486232x_3^{33} \\
& +462728840082228859334694943477743198307650012420374125365389249712651768x_3^{32} \\
& -131776588232594972255777649206567320338061471477811725806702054151290252x_3^{31} \\
& +31472043180361439617980833132544850110943883369080890722278786629107628x_3^{30} \\
& -5961664481968065504823382003538804229336739136240908717770483137477456x_3^{29} \\
& +722351769246397427865270518182475924172222357378144113248455430696656x_3^{28} \\
& +93878343461212773918586759022334368944010651758103940471975728130612x_3^{27} \\
& -95592681800341241277296233300832558302214353663993838838577883834868x_3^{26} \\
& +41634790191131063210888629765583357020845625933576181588238778182552x_3^{25} \\
& -13936295068064242654729392915446150762330722439463395349506372368488x_3^{24} \\
& +4045182362993096546784545052702282647971953511355346978628439483220x_3^{23} \\
& -1062701894734956175602327175315458674030959500282038965423982676924x_3^{22} \\
& +256459649784050971207284569212441996943675040503103180406710637212x_3^{21} \\
& -57709734724845648912090174522151135714278853267038668156157868992x_3^{20} \\
& +12153005083455119529122284311415304647462102060888120578436682307x_3^{19} \\
& -2406401626414324343746563575869808824782931044900643158950599971x_3^{18} \\
& +449105840481193161884750567257028703367516412585554400171866890x_3^{17} \\
& -78996265368429079118013549058436508446509452503993511834925650x_3^{16} \\
& +13109093052218561111137671541747428560864068581509706530641689x_3^{15} \\
& -2049218622402956644350490453379175367227864675085518185017233x_3^{14} \\
& +301303415905142916449843588495664606433759693923619398291732x_3^{13} \\
& -41580761868772328901842590178904706369026472084206008451876x_3^{12} \\
& +5365113040262943479433997705054180067036308260891222155505x_3^{11} \\
& -644693829313954760172498533221854992519656915601541723705x_3^{10} \\
& +71731828298864812802268603899355624836434020184867743210x_3^9 \\
& -7334795513210803706517436006266060058370643492181530930x_3^8 \\
& +682989728878875810684537346120890928588745523322856055x_3^7 \\
& -57172697972619449741387104525586264686698343949422455x_3^6 \\
& +4226158290506747020131443419536269019866825889091500x_3^5 \\
& -269121764902623042560625370550056308979867340869800x_3^4 \\
& +14198857298392070342554145297476835831442968012100x_3^3 \\
& -578321988619101115121213409365880613402609688700x_3^2 \\
& +15961340415131975059297267606639156809924912000x_3 \\
& -220720851022013939349538517195371169708198400
\end{aligned}$$

Solving $h_1(x_3) = 0$ numerically, we get five positive and four negative solutions, which are given approximately by $x_3 \approx 0.2205104$, $x_3 \approx 0.072293790$, $x_3 \approx 0.11492789$, $x_3 \approx 1.5437649$, $x_3 \approx 1.972915$ and $x_3 \approx -0.30929231$, $x_3 \approx -0.30556009$, $x_3 \approx -0.44683455$, $x_3 \approx -0.39992522$. Furthermore, we take the following real solutions of the system $\{g_0 = 0, g_1 = 0, g_2 = 0, g_3 = 0, g_4 = 0, h_1(x_3) = 0\}$ with $u_0 u_1 u_2 x_2 x_3 \neq 0$:

$$\begin{aligned}
& \{u_0 \approx 1.0725485, u_1 \approx 0.045012693, u_2 \approx 0.31014510, x_3 \approx 0.2205104, x_2 \approx 0.75504389\}, \\
& \{u_0 \approx 0.073985052, u_1 \approx 0.067061174, u_2 \approx 0.30836161, x_3 \approx 0.072293790, x_2 \approx 1.0001177\}, \\
& \{u_0 \approx 0.12074935, u_1 \approx 0.096083894, u_2 \approx 1.0975908, x_3 \approx 0.11492789, x_2 \approx 1.0004421\}, \\
& \{u_0 \approx 1.5886788, u_1 \approx 1.3434091, u_2 \approx 0.37367555, x_3 \approx 1.5437649, x_2 \approx 1.1394129\}, \\
& \{u_0 \approx 2.0094538, u_1 \approx 0.057685208, u_2 \approx 0.39443471, x_3 \approx 1.9729151, x_2 \approx 1.4757000\}
\end{aligned}$$

and

$$\begin{aligned}
& \{u_0 \approx 0.57479844, u_1 \approx 0.091481076, u_2 \approx 1.0446277, x_3 \approx -0.30929231, x_2 \approx 2.2251780\}, \\
& \{u_0 \approx 0.56438862, u_1 \approx 0.57295621, u_2 \approx 0.98344796, x_3 \approx -0.30556009, x_2 \approx 2.3226977\}, \\
& \{u_0 \approx 1.1366881, u_1 \approx 0.71052398, u_2 \approx 0.50096966, x_3 \approx -0.44683455, x_2 \approx 2.5636731\}, \\
& \{u_0 \approx 0.95640165, u_1 \approx 0.80160322, u_2 \approx 0.54149306, x_3 \approx -0.39992522, x_2 \approx 2.5715323\}.
\end{aligned}$$

Thus we obtain five Einstein metrics which according to Proposition 5.5, are non-naturally reductive. In particular, a computation of the related scale invariants shows that these five metrics are non-isometric each other.

For $14863x_3^3 - 23537x_3^2 + 3841x_3 - 159 = 0$, we see that $x_2 = 1, u_0 = u_1 = x_3$ and $44589x_3^2 - 65894x_3 + 2756u_2 + 3573 = 0$ are solutions of the system of equations $\{g_0 = 0, g_1 = 0, g_2 = 0, g_3 = 0, g_4 = 0\}$. Thus, approximately we obtain the following solutions:

$$\begin{aligned} \{u_0 = u_1 = x_3 \approx 0.070481512, u_2 \approx 0.30834779, x_2 = 1\}, \\ \{u_0 = u_1 = x_3 \approx 1.4050939, u_2 \approx 0.35656500, x_2 = 1\}, \\ \{u_0 = u_1 = x_3 \approx 0.10802147, u_2 \approx 1.0974869, x_2 = 1\}. \end{aligned}$$

For $x_3 = 1$, we see that $(x_2 - 1)(864x_2^3 - 1676x_2^2 + 973x_2 - 177) = 0, u_0 = u_2 = x_2$ and $59u_1 - 3456x_2^3 + 10160x_2^2 - 9003x_2 + 2240 = 0$. Thus, in this case the solutions approximately have the form

$$\begin{aligned} \{u_0 = u_2 = x_2 \approx 0.41888766, u_1 \approx 0.042734087, x_3 = 1\}, \\ \{u_0 = u_2 = x_2 \approx 0.46172446, u_1 \approx 1.5439133, x_3 = 1\}, \\ \{u_0 = u_2 = x_2 \approx 1.0592027, u_1 \approx 0.072250884, x_3 = 1\} \end{aligned}$$

and $u_0 = u_1 = x_2 = x_3 = 1$. According to Proposition 5.5, these values define naturally reductive Einstein metrics.

Case of $E_7(3)$

$$(5.11) \quad \left\{ \begin{aligned} g_0 &= 4u_0u_1x_2^2x_3^2 + 6u_0u_1x_2^2 + 8u_0u_1x_3^2 - 9u_1^2x_2^2x_3^2 - u_1^2x_2^2 - 3u_1^2x_3^2 - 5x_2^2x_3^2 = 0, \\ g_1 &= 9u_1^2u_2x_2^2x_3^2 + u_1^2u_2x_2^2 + 3u_1^2u_2x_3^2 - 10u_1u_2^2x_2^2x_3^2 - 5u_1u_2^2x_3^2 - 3u_1x_2^2x_3^2 \\ &\quad + 5u_2x_2^2x_3^2 = 0, \\ g_2 &= 2u_0u_2x_2^2x_3 + 108u_1u_2x_2^2x_3 + 190u_2^2x_2^2x_3 + 75u_2^2x_3 + 90u_2x_2^3x_3 + 30u_2x_2^3 \\ &\quad - 540u_2x_2^2x_3 + 30u_2x_2x_3^2 - 30u_2x_2 + 45x_2^2x_3 = 0, \\ g_3 &= -u_0x_2^2x_3 + 4u_0x_3 - 54u_1x_2^2x_3 + 36u_1x_3 - 20u_2x_2^2x_3 + 20u_2x_3 - 90x_2^3x_3 - 45x_2^3 \\ &\quad + 270x_2^2x_3 + 15x_2x_3^2 - 180x_2x_3 + 45x_2 = 0, \\ g_4 &= 9u_0x_2^2 - 4u_0x_3^2 + 36u_1x_2^2 - 36u_1x_3^2 - 20u_2x_3^2 + 45x_2^3x_3^2 + 120x_2^3x_3 - 270x_2^2x_3 \\ &\quad - 120x_2x_3^3 + 180x_2x_3^2 + 60x_2x_3 = 0. \end{aligned} \right.$$

We consider the polynomial ring $R = \mathbb{Q}[z, u_0, u_1, u_2, x_2, x_3]$ and an ideal I generated by polynomials $\{g_0, g_1, g_2, g_3, g_4, z u_0 u_1 u_2 x_2 x_3 - 1\}$. We take a lexicographic order $>$ with $z > u_0 > u_1 > u_2 > x_2 > x_3$ for a monomial ordering on R . Then, by the aid of computer, we see that a Gröbner basis for the ideal I contains a polynomial of x_3 given by $(x_3 - 1)(5632x_3^3 - 9488x_3^2 + 3933x_3 - 477)h_1(x_3)$, where $h_1(x_3)$ is a polynomial of degree 119 given by

$$\begin{aligned} h_1(x_3) &= 5317991353240733131727570427468867047230653060663383949312000000x_3^{119} \\ &\quad - 98019036075297848183056077552432780274599684871346934691921920000x_3^{118} \\ &\quad + 1098759293323246546512110892440119473621419358215417540389109760000x_3^{117} \\ &\quad - 9305718974821085677811148533918240248548281811944475222438051840000x_3^{116} \\ &\quad + 65113268802402200478525426791012973944464656086732606877047193600000x_3^{115} \\ &\quad - 394046712276679250169937971915477424725853689736022406260651458560000x_3^{114} \\ &\quad + 2120477184719184687909529467574362503309040056101340761096036810752000x_3^{113} \\ &\quad - 10334275693315162730742501425160945657985705715529777725552751280128000x_3^{112} \\ &\quad + 46202883626048272893214883998659314525038880033265154243257408946176000x_3^{111} \\ &\quad - 191297437298409715833973441970431226167739110188661910161889358249984000x_3^{110} \\ &\quad + 738780827716860155253065617261688708737616255470467254037109394531942400x_3^{109} \\ &\quad - 2676181837866157465124121869252517828887974363206276139033897785517670400x_3^{108} \\ &\quad + 9133426483052267815004033114279867587137539131902245605998876572093644800x_3^{107} \\ &\quad - 29472676795039732970734547702915016051055469216731276093351250213864755200x_3^{106} \\ &\quad + 90184928909873771799432761661136216708184513107610597609746891377263136000x_3^{105} \end{aligned}$$

$-262307357426217645783615015266291229953598598109678467351599564732054385920x_3^{104}$
 $+726607074049122059949903497786269863005806535278198855839594575897076134400x_3^{103}$
 $-1920000384806180026768867421031537805803949214018610412439630758504547658240x_3^{102}$
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 $+6294369642269549673426137151386593173604138265575910629636206811805707511761437x_3^{81}$
 $+13818211675329783251259102108459155176182048091914822307750576223359914637453455x_3^{80}$
 $-37746690309720320985994632077313901348434569764528336593925529654080198675646244x_3^{79}$
 $+61753673833125499831493137549382712686215192881835369866733354263253711754859516x_3^{78}$
 $-80854547668572429281210675119560892341208614027928965716233593125981815435023888x_3^{77}$
 $+89812864504052969401079860636375036763636864288507524282286470644623557615081560x_3^{76}$
 $-84535429159155233862550289641436464123003887379615811860052338807579720883023478x_3^{75}$
 $+63510492448268670372687644118857216784764228069599171336461180087553652873279182x_3^{74}$
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$$\begin{aligned}
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& +475504463177526413957760735598869270023707338832121518609174521427627560x_3^{22} \\
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& -8026521807779078099496238258304727304087936835093634106534385898x_3^6 \\
& +1081478431569654626009258053539930201185578706540437961929852917x_3^5 \\
& -118566734244541430886739082169060500770838134263466432240586949x_3^4 \\
& +10256500407063673515218118277406679603598463165413692852750954x_3^3 \\
& -661996321847218796068287208154089929196932463764640942695594x_3^2 \\
& +28572142588965250828016623923248449622787682754546590053837x_3 \\
& -623744015652591601670205734762122870831994162006221990413.
\end{aligned}$$

Solving $h_1(x_3) = 0$ numerically, we obtain seven real solutions which are given by $x_3 \approx 1.1800573$, $x_3 \approx 0.49280351$, $x_3 \approx 1.1060677$, $x_3 \approx 1.3849054$, $x_3 \approx 2.4753269$ (we state only the positive). As a consequence, real solutions of the system of equations $\{g_0 = 0, g_1 = 0, g_2 = 0, g_3 = 0, g_4 = 0, h_1(x_3) = 0\}$ with $u_0 u_1 u_2 x_2 x_3 \neq 0$ are of the form

$$\begin{aligned}
& \{u_0 \approx 0.30587680, u_1 \approx 0.23162043, u_2 \approx 0.11719295, x_2 \approx 1.0035307, x_3 \approx 0.27827971\}, \\
& \{u_0 \approx 0.43465453, u_1 \approx 0.27733727, u_2 \approx 1.4182653, x_2 \approx 1.0086185, x_3 \approx 0.37945991\}, \\
& \{u_0 \approx 0.33445150, u_1 \approx 0.23695076, u_2 \approx 0.36978513, x_2 \approx 0.31241976, x_3 \approx 1.0008636\}, \\
& \{u_0 \approx 0.28679936, u_1 \approx 0.36605764, u_2 \approx 0.14958786, x_2 \approx 0.28763468, x_3 \approx 1.0026185\}, \\
& \{u_0 \approx 0.77541704, u_1 \approx 0.19715742, u_2 \approx 0.11437270, x_2 \approx 0.52666358, x_3 \approx 1.0826430\}, \\
& \{u_0 \approx 1.5820396, u_1 \approx 0.30622692, u_2 \approx 1.3221125, x_2 \approx 1.2303151, x_3 \approx 1.3552648\}, \\
& \{u_0 \approx 2.3846395, u_1 \approx 0.30253103, u_2 \approx 0.17015362, x_2 \approx 1.6249173, x_3 \approx 2.2246116\}.
\end{aligned}$$

Thus, we obtain seven Einstein metrics which are non-naturally reductive by Proposition 5.5. In particular, these seven metrics are non-isometric each other and this follows after a computation of the corresponding scale invariants.

For $(x_3 - 1)(5632x_3^3 - 9488x_3^2 + 3933x_3 - 477) = 0$ For $5632x_3^3 - 9488x_3^2 + 3933x_3 - 477 = 0$, we see that $x_2 = 1$, $u_0 = u_1 = x_3$ and $1408x_3^2 - 1948x_3 + 53u_2 + 387 = 0$ for solutions of the system of equations $\{g_0 = 0, g_1 = 0, g_2 = 0, g_3 = 0, g_4 = 0\}$. Hence, in this case we conclude that the following parameters define solutions of the homogeneous Einstein equation:

$$\begin{aligned} \{u_0 = u_1 = x_3 \approx 0.24536236, u_2 \approx 0.11699044, x_2 = 1\}, \\ \{u_0 = u_1 = x_3 \approx 0.30406198, u_2 \approx 1.4176970, x_2 = 1\}, \\ \{u_0 = u_1 = x_3 \approx 1.1352348, u_2 \approx 0.18615305, x_2 = 1\}. \end{aligned}$$

For $x_3 = 1$, we see that $(x_2 - 1)(7x_2 - 2) = 0$, $u_0 = u_1 = u_2 = x_2$. Thus we obtain a solution, given by

$$\{u_0 = u_1 = u_2 = x_2 = 2/7, x_3 = 1\},$$

and $u_0 = u_1 = x_2 = x_3 = 1$. Using Proposition 5.5 we deduce that the induced left-invariant Einstein metric are naturally reductive.

6. LEFT-INVARIANT NON-NATURALLY REDUCTIVE EINSTEIN METRICS ON LIE GROUPS OF TYPE $III_b(3)$

6.1. The Lie groups $E_6(3)$. In this final section we examine the Lie group $E_6(3)$, which is the unique compact simple Lie group $G = G(i_o)$ of Type $III_b(3)$, see Theorem 3.3. Let us denote its Lie algebra by \mathfrak{g} . Consider the orthogonal decomposition

$$(6.1) \quad \mathfrak{g} = \mathfrak{k}_0 \oplus \mathfrak{k}_1 \oplus \mathfrak{k}_2 \oplus \mathfrak{k}_3 \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3 = \mathfrak{m}_0 \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3 \oplus \mathfrak{m}_4 \oplus \mathfrak{m}_5 \oplus \mathfrak{m}_6.$$

with $\mathfrak{k}_0 \cong \mathfrak{u}_1$, $\mathfrak{k}_1 \cong \mathfrak{su}_2$, $\mathfrak{k}_2 \cong \mathfrak{su}_3 \cong \mathfrak{k}_3$. Hence, and according to Table 2, it is $d_0 = 1$, $d_1 = 3$, $d_2 = d_3 = 8$, $d_4 = 36$, $d_5 = 18$ and $d_6 = 4$. A left-invariant metric on E_6 is given by

$$(6.2) \quad \begin{aligned} \langle \cdot, \cdot \rangle &= u_0 \cdot B|_{\mathfrak{k}_0} + u_1 \cdot B|_{\mathfrak{k}_1} + u_2 \cdot B|_{\mathfrak{k}_2} + u_3 \cdot B|_{\mathfrak{k}_3} + x_1 \cdot B|_{\mathfrak{p}_1} + x_2 \cdot B|_{\mathfrak{p}_2} + x_3 \cdot B|_{\mathfrak{p}_3} \\ &= y_0 \cdot B|_{\mathfrak{m}_0} + y_1 \cdot B|_{\mathfrak{m}_1} + y_2 \cdot B|_{\mathfrak{m}_2} + y_3 \cdot B|_{\mathfrak{m}_3} + y_4 \cdot B|_{\mathfrak{m}_4} + y_5 \cdot B|_{\mathfrak{m}_5} + y_6 \cdot B|_{\mathfrak{m}_6}, \end{aligned}$$

for some positive numbers $u_i, x_j, y_m \in \mathbb{R}_+$. This metric is also $\text{Ad}(K)$ -invariant and since $\mathfrak{m}_i \not\cong \mathfrak{m}_j$ for any $3 \leq i \neq j \leq 6$, all G -invariant metrics on the base space $M = G/K$ are a multiple of

$$(\cdot, \cdot) = x_1 \cdot B|_{\mathfrak{p}_1} + x_2 \cdot B|_{\mathfrak{p}_2} + x_3 \cdot B|_{\mathfrak{p}_3} = y_4 \cdot B|_{\mathfrak{m}_4} + y_5 \cdot B|_{\mathfrak{m}_5} + y_6 \cdot B|_{\mathfrak{m}_6}.$$

In a similar way with Proposition 5.1, we conclude that the non-zero triples A_{ijk} ($0 \leq i, j, k \leq 6$) associated to the reductive decomposition (6.1) and the left-invariant metric on E_6 given by (6.2), are the following (up to permutation of indices):

$$A_{044}, A_{055}, A_{066}, A_{111}, A_{144}, A_{166}, A_{222}, A_{244}, A_{255}, A_{333}, A_{344}, A_{355}, A_{445}, A_{456}.$$

In particular, it is easy to see that $A_{155} = A_{266} = A_{366} = 0$.

Remark 6.1. In the reductive decomposition (6.1) it is $\mathfrak{k}_2 \cong \mathfrak{su}_3 \cong \mathfrak{k}_3$. This isomorphism does not effect on the behaviour of the Ricci tensor $\text{Ric}_{\langle \cdot, \cdot \rangle}$ corresponding left-invariant metric $\langle \cdot, \cdot \rangle$. In particular, by using root vectors corresponding to \mathfrak{k}_2 and \mathfrak{k}_3 , it follows that $\text{Ric}_{\langle \cdot, \cdot \rangle}(\mathfrak{k}_2, \mathfrak{k}_3) = 0$, hence $\text{Ric}_{\langle \cdot, \cdot \rangle}$ is still diagonal.

6.2. The Ricci tensor and the structure constants. Let us apply Lemma 2.2 to get a first version of the Ricci tensor in terms of the parameters of $\langle \cdot, \cdot \rangle$, the dimensions d_i and the non-zero triples A_{ijk} .

Proposition 6.2. *The components r_i of the Ricci tensor $\text{Ric}_{\langle \cdot, \cdot \rangle}$ associated to the left-invariant metric $\langle \cdot, \cdot \rangle$ on E_6 described by (6.2), are given by*

$$\left\{ \begin{aligned} r_0 &= \frac{u_0}{4d_0} \left(\frac{A_{044}}{x_1^2} + \frac{A_{055}}{x_2^2} + \frac{A_{066}}{x_3^2} \right), \quad r_1 = \frac{A_{111}}{4d_1} \cdot \frac{1}{u_1} + \frac{u_1}{4d_1} \left(\frac{A_{144}}{x_1^2} + \frac{A_{166}}{x_3^2} \right), \\ r_2 &= \frac{A_{222}}{4d_2} \cdot \frac{1}{u_2} + \frac{u_2}{4d_2} \left(\frac{A_{244}}{x_1^2} + \frac{A_{255}}{x_2^2} \right), \quad r_3 = \frac{A_{333}}{4d_3} \cdot \frac{1}{u_3} + \frac{u_3}{4d_3} \left(\frac{A_{344}}{x_1^2} + \frac{A_{355}}{x_2^2} \right), \\ r_4 &= \frac{1}{2x_1} - \frac{1}{2d_4x_1^2} \left(u_0 \cdot A_{044} + u_1 \cdot A_{144} + u_2 \cdot A_{244} + u_3 \cdot A_{344} + x_2 \cdot A_{445} \right) + \frac{A_{456}}{2d_4} \left(\frac{x_1}{x_2x_3} - \frac{x_2}{x_1x_3} - \frac{x_3}{x_1x_2} \right), \\ r_5 &= \frac{1}{2x_2} - \frac{1}{2d_5x_2^2} \left(u_0 \cdot A_{055} + u_2 \cdot A_{255} + u_3 \cdot A_{355} \right) + \frac{A_{445}}{4d_5} \left(\frac{x_2}{x_1^2} - \frac{2}{x_2} \right) + \frac{A_{456}}{2d_5} \left(\frac{x_2}{x_1x_3} - \frac{x_1}{x_2x_3} - \frac{x_3}{x_1x_2} \right), \\ r_6 &= \frac{1}{2x_3} - \frac{1}{2d_6} \cdot \frac{1}{x_3^2} \left(u_0 \cdot A_{066} + u_1 \cdot A_{166} \right) + \frac{A_{456}}{2d_5} \left(\frac{x_3}{x_1x_2} - \frac{x_1}{x_2x_3} - \frac{x_2}{x_1x_3} \right). \end{aligned} \right.$$

We need now the values of the non-zero A_{ijk} . These are described by the following lemma.

Lemma 6.3. *For the reductive decomposition (6.1) and for the left-invariant metric $\langle \cdot, \cdot \rangle$ on the Lie group $E_6 = E_6(3)$, the non-zero A_{ijk} are given explicitly as follows:*

$$\begin{aligned} A_{044} &= 1/4, \quad A_{055} = 1/2, \quad A_{066} = 1/4, \quad A_{111} = 1/2, \quad A_{144} = 9/4, \quad A_{166} = 1/4, \\ A_{222} &= A_{255} = A_{333} = A_{355} = 2, \quad A_{244} = A_{344} = 4, \quad A_{445} = 6, \quad A_{456} = 3/2. \end{aligned}$$

Proof. The computation of A_{445} and A_{456} is based on the unique Kähler-Einstein metric $y_4 = 1, y_5 = 2, y_6 = 3$ that $M = G/K = E_6/(U_1 \times SU_2 \times SU_3 \times SU_3)$ admits. Thus, the system which defines the Killing metric $y_i = 1$ ($i = 0, \dots, 6$), consists now of six equations and 12 unknowns. For the construction of more equations, let us consider first the twistor fibration of $M = G/K$ over an irreducible symmetric space. Set

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}, \quad \mathfrak{h} := \mathfrak{h}_1 \oplus \mathfrak{h}_2, \quad \mathfrak{h}_1 := \mathfrak{k}_0 \oplus \mathfrak{k}_2 \oplus \mathfrak{k}_3 \oplus \mathfrak{p}_2, \quad \mathfrak{h}_2 := \mathfrak{k}_1 \cong \mathfrak{su}_2, \quad \mathfrak{n} := \mathfrak{p}_1 \oplus \mathfrak{p}_3.$$

This is a symmetric reductive decomposition of \mathfrak{g} ; for dimensional reasons we take $\mathfrak{h}_1 \cong \mathfrak{su}_6$ and the corresponding irreducible symmetric space G/H is the coset $E_6/(SU_6 \times SU_2)$. Consider a left-invariant metric on E_6 , given by $\langle \cdot, \cdot \rangle = v_1 \cdot B|_{\mathfrak{h}_1} + v_2 \cdot B|_{\mathfrak{h}_2} + v_3 \cdot B|_{\mathfrak{n}}$, for some $v_1, v_2, v_3 \in \mathbb{R}_+$. For $v_1 = u_0 = u_2 = u_3 = x_2$, $v_2 = u_1$, $v_3 = x_1 = x_3$ this metric coincides with $\langle \cdot, \cdot \rangle$ and the same holds for the corresponding Ricci tensors. Hence we get the following equations:

$$\begin{aligned} d_0 A_{244} - d_2(A_{044} + A_{066}) &= d_2 A_{344} - d_3 A_{244}, \\ -d_2 A_{344} + d_3 A_{244} &= d_5 A_{344}(d_4 + 4d_5 + 9d_6) - 4d_3 d_4 d_6 - d_3 d_4 d_5 - d_3 d_5 d_6, \\ d_0(A_{222} + A_{255}) - d_2 A_{055} &= d_2(A_{333} + A_{355}) - d_3(A_{222} + A_{255}), \\ -d_2(A_{333} + A_{355}) + d_3(A_{222} + A_{255}) &= (d_4 + 4d_5 + 9d_6)(2d_3(A_{055} + A_{255} + A_{355}) + d_5(A_{333} + A_{355})) \\ &\quad - 8d_5^2 d_3 + 8d_3 d_4 d_6 - 16d_3 d_5 d_6, \\ 0 &= (d_4 + 4d_5 + 9d_6)(d_4 A_{166} - d_6 A_{144}), \\ 0 &= d_4 A_{066}(d_4 + 4d_5 + 9d_6) - d_6(A_{044} + A_{244} + A_{344})(d_4 + 4d_5) \\ &\quad - 9d_6^2(A_{044} + A_{244} + A_{344}) - 3d_4 d_6^2 + d_4^2 d_6. \end{aligned}$$

Set now

$$\mathfrak{g} = \mathfrak{q} \oplus \mathfrak{r}, \quad \mathfrak{q} := \mathfrak{q}_1 \oplus \mathfrak{q}_2 \oplus \mathfrak{q}_3, \quad \mathfrak{q}_1 := \mathfrak{k}_0 \oplus \mathfrak{k}_1 \oplus \mathfrak{p}_3, \quad \mathfrak{q}_2 := \mathfrak{k}_2, \quad \mathfrak{q}_3 := \mathfrak{k}_3, \quad \mathfrak{r} := \mathfrak{p}_1 \oplus \mathfrak{p}_2.$$

It follows that $\mathfrak{q}_i \cong \mathfrak{su}_3$ for any $i = 1, 2, 3$ and $[\mathfrak{q}, \mathfrak{q}] \subset \mathfrak{q}$, $[\mathfrak{q}, \mathfrak{r}] \subset \mathfrak{r}$. Since $\mathfrak{k} \subset \mathfrak{q}$, this defines the fibration

$$\mathbb{C}P^2 = SU_3/U_2 \rightarrow E_6/(U_1 \times SU_2 \times SU_3 \times SU_3) \rightarrow E_6/(SU_3 \times SU_3 \times SU_3),$$

where the base space $G/Q \cong E_6/(SU_3 \times SU_3 \times SU_3)$ is isotropy irreducible, see [B]. By a similar procedure as before and after considering a new left-invariant metric on E_6 , we get that

$$\begin{aligned} -d_0 A_{144} + d_1(A_{044} + A_{055}) &= d_6 A_{144}(d_4 + 4d_5 + 9d_6) - 2d_1 d_6(d_4 + d_5), \\ -d_0(A_{111} + A_{166}) + d_1 A_{066} &= 2d_1(A_{066} + A_{166})(d_4 + 4d_5 + 9d_6) + d_6(A_{111} + A_{166})(d_4 + 4d_5 + 9d_6) \\ &\quad + 2d_1 d_4 d_6 - 4d_1 d_5 d_6 - 18d_1 d_6^2. \end{aligned}$$

Now, a combination of these equations together with the system defined by the Killing metric, shows that $A_{044} = A_{066} = A_{166} = 1/4$, $A_{055} = A_{111} = 1/2$, $A_{144} = 9/4$, $A_{244} = A_{344} = 4$ and

$$A_{222} = 4 - A_{255}, \quad A_{333} = A_{255}, \quad A_{355} = 4 - A_{255}.$$

However, it is $A_{222} = c \cdot \dim \mathfrak{su}_3$, where $c = B_{SU_3}/B_{E_6} = 4/24$. Thus, $A_{222} = 2$ and we also get $A_{255} = 2 = A_{333} = A_{355}$. In fact, we can verify the values of A_{111}, A_{333} as follows: $A_{111} = (B_{SU_2}/B_{E_6}) \cdot \dim \mathfrak{su}_2 = 1/2$ and $A_{333} = (B_{SU_3}/B_{E_6}) \cdot \dim \mathfrak{su}_3 = 2 (= A_{222})$. \square

6.3. Naturally reductive metrics. For a Lie group $G \cong G(i_o)$ of Type $III_b(3)$, left-invariant metrics on $G = E_6(3)$ which are $\text{Ad}(K)$ -invariant are given by

$$(6.3) \quad \langle \cdot, \cdot \rangle = u_0 \cdot B|_{\mathfrak{k}_0} + u_1 \cdot B|_{\mathfrak{k}_1} + u_2 \cdot B|_{\mathfrak{k}_2} + u_3 \cdot B|_{\mathfrak{k}_3} + x_1 \cdot B|_{\mathfrak{p}_1} + x_2 \cdot B|_{\mathfrak{p}_2} + x_3 \cdot B|_{\mathfrak{p}_3}.$$

Proposition 6.4. *If a left invariant metric $\langle \cdot, \cdot \rangle$ of the form (6.3) on $G = E_6(3)$ of Type $III_b(3)$ is naturally reductive with respect to $G \times L$ for some closed subgroup L of G , then one of the following holds:*

- (1) $u_0 = u_2 = u_3 = x_2$, $x_1 = x_3$ (2) $u_0 = u_1 = x_3$, $x_1 = x_2$ (3) $x_1 = x_2 = x_3$.

Conversely, if one of the conditions (1), (2), (3) holds, then the metric $\langle \cdot, \cdot \rangle$ of the form (6.3) is naturally reductive with respect to $G \times L$ for some closed subgroup L of G .

Proof. Let \mathfrak{l} be the Lie algebra of L . Then we have either $\mathfrak{l} \subset \mathfrak{k}$ or $\mathfrak{l} \not\subset \mathfrak{k}$. First we consider the case of $\mathfrak{l} \not\subset \mathfrak{k}$. Let \mathfrak{h} be the subalgebra of \mathfrak{g} generated by \mathfrak{l} and \mathfrak{k} . Since $\mathfrak{g} = \mathfrak{k}_0 \oplus \mathfrak{k}_1 \oplus \mathfrak{k}_2 \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3$ is an irreducible decomposition as $\text{Ad}(K)$ -modules, we see that the Lie algebra \mathfrak{h} contains at least one of $\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3$. Let us start with the case $\mathfrak{p}_1 \subset \mathfrak{h}$. In this case, we get $[\mathfrak{p}_1, \mathfrak{p}_1] \cap \mathfrak{p}_2 \neq \{0\}$ and hence $\mathfrak{p}_2 \subset \mathfrak{h}$. Notice also that $[\mathfrak{p}_1, \mathfrak{p}_2] \cap \mathfrak{p}_3 \neq \{0\}$, i.e. \mathfrak{h} contains \mathfrak{p}_3 as well. Therefore, $\mathfrak{h} = \mathfrak{g}$ coincide, and the $\text{Ad}(L)$ -invariant metric $\langle \cdot, \cdot \rangle$ of the form (6.3) is bi-invariant. Now if $\mathfrak{p}_2 \subset \mathfrak{h}$, then $\mathfrak{h} \supset \mathfrak{k} \oplus \mathfrak{p}_2$. If $\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{p}_2$, then $(\mathfrak{h}, \mathfrak{p}_1 \oplus \mathfrak{p}_3)$ is a symmetric pair. Thus, the metric $\langle \cdot, \cdot \rangle$ of the form (6.3) satisfies $u_0 = u_2 = u_3 = x_2, x_1 = x_3$. If $\mathfrak{h} \neq \mathfrak{k} \oplus \mathfrak{p}_2$, then it must be $\mathfrak{h} \cap \mathfrak{p}_1 \neq \{0\}$ or $\mathfrak{h} \cap \mathfrak{p}_3 \neq \{0\}$ and thus $\mathfrak{h} \supset \mathfrak{p}_1$, or $\mathfrak{h} \supset \mathfrak{p}_3$. Hence, we conclude again $\mathfrak{h} = \mathfrak{g}$ and the $\text{Ad}(L)$ -invariant metric $\langle \cdot, \cdot \rangle$ of the form (6.3) must be bi-invariant. Finally, if \mathfrak{h} contains \mathfrak{p}_3 , then $\mathfrak{h} \supset \mathfrak{k} \oplus \mathfrak{p}_3$. If $\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{p}_3$, then \mathfrak{h} is a semi-simple Lie algebra and $\mathfrak{p}_1 \oplus \mathfrak{p}_2$ is an irreducible $\text{Ad}(H)$ -module. Hence, the metric $\langle \cdot, \cdot \rangle$ defined by (6.3) satisfies the conditions $u_0 = u_1 = x_3, x_1 = x_2$. If $\mathfrak{h} \neq \mathfrak{k} \oplus \mathfrak{p}_3$, we similarly conclude that $\mathfrak{h} \cap \mathfrak{p}_1 \neq \{0\}$ or $\mathfrak{h} \cap \mathfrak{p}_2 \neq \{0\}$. Thus it must be $\mathfrak{h} \supset \mathfrak{p}_1$, or $\mathfrak{h} \supset \mathfrak{p}_2$. In the same way, we obtain the identification $\mathfrak{h} = \mathfrak{g}$ and the $\text{Ad}(L)$ -invariant metric $\langle \cdot, \cdot \rangle$ given by (6.3) is bi-invariant.

Now, consider the case $\mathfrak{l} \subset \mathfrak{k}$. Since the orthogonal complement \mathfrak{l}^\perp of \mathfrak{l} with respect to B contains the orthogonal complement \mathfrak{k}^\perp of \mathfrak{k} , we see that $\mathfrak{l}^\perp \supset \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3$. Since the invariant metric $\langle \cdot, \cdot \rangle$ is naturally reductive with respect to $G \times L$, it follows that $x_1 = x_2 = x_3$, by Theorem 2.1.

Conversely, if the condition (1) holds, then according to Theorem 2.1 the metric $\langle \cdot, \cdot \rangle$ given by (6.3) is naturally reductive with respect to $G \times L$, where $\mathfrak{l} = \mathfrak{k} \oplus \mathfrak{p}_2$. If the condition (2) holds, then the metric given by (6.3) is naturally reductive with respect to $G \times L$ where $\mathfrak{l} = \mathfrak{k} \oplus \mathfrak{p}_3$. Finally, considering the condition (3), the metric $\langle \cdot, \cdot \rangle$ defined by (6.3) is naturally reductive with respect to $G \times K$. This completes the proof. \square

6.4. The homogeneous Einstein equation. Due to Proposition 6.2 and Lemma 6.3, we are able now to describe explicitly the Einstein equation on E_6 with respect to the left-invariant metric $\langle \cdot, \cdot \rangle$ given by (6.3). This has the form (we normalise $\langle \cdot, \cdot \rangle$ by setting $x_1 = 1$):

$$(6.4) \quad \begin{cases} g_0 = 3u_0u_1x_2^2x_3^2 + 3u_0u_1x_2^2 + 6u_0u_1x_3^2 - 9u_1^2x_2^2x_3^2 - u_1^2x_2^2 - 2x_2^2x_3^2 = 0, \\ g_1 = 9u_1^2u_2x_2^2x_3^2 + u_1^2u_2x_2^2 - 6u_1u_2^2x_2^2x_3^2 - 3u_1u_2^2x_3^2 - 3u_1x_2^2x_3^2 + 2u_2x_2^2x_3^2 = 0, \\ g_2 = (u_2 - u_3)(2u_2u_3x_2^2 + u_2u_3 - x_2^2) = 0, \\ g_3 = u_0u_3x_2^2x_3 + 9u_1u_3x_2^2x_3 + 16u_2u_3x_2^2x_3 + 52u_3^2x_2^2x_3 + 18u_3^2x_3 + 24u_3x_2^3x_3 + 6u_3x_2^3 \\ \quad - 144u_3x_2^2x_3 + 6u_3x_2x_3^2 - 6u_3x_2 + 18x_2^2x_3 = 0, \\ g_4 = -u_0x_2^2x_3 + 4u_0x_3 - 9u_1x_2^2x_3 - 16u_2x_2^2x_3 + 16u_2x_3 - 16u_3x_2^2x_3 + 16u_3x_3 - 48x_2^3x_3 \\ \quad - 18x_2^3 + 144x_2^2x_3 + 6x_2x_3^2 - 96x_2x_3 + 18x_2 = 0, \\ g_5 = 9u_0x_2^2 - 4u_0x_3^2 + 9u_1x_2^2 - 16u_2x_3^2 - 16u_3x_3^2 + 24x_2^3x_3^2 + 66x_2^3x_3 - 144x_2^2x_3 \\ \quad - 66x_2x_3^3 + 96x_2x_3^2 + 42x_2x_3 = 0. \end{cases}$$

Hence, here we need to separate our study in two cases: $u_2 = u_3$ and $u_2 \neq u_3$.

Case of $u_2 = u_3$.

Consider the polynomial ring $R = \mathbb{Q}[z, u_0, u_1, u_2, u_3, x_2, x_3]$ and the ideal I , generated by polynomials $\{g_0, g_1, g_2, u_2 - u_3, g_4, g_5, z u_0 u_1 u_2 u_3 x_2 x_3 - 1\}$. We take a lexicographic ordering $>$, with $z > u_0 > u_1 > u_2 > u_3 > x_2 > x_3$ for a monomial ordering on R . Then, a Gröbner basis for the ideal I , contains a polynomial of x_3 given by $(x_3 - 1)(17x_3 - 3)(105x_3^2 - 180x_3 + 43)h_1(x_3)$, where $h_1(x_3)$ is a polynomial of degree 101 explicitly defined as follows:

$$\begin{aligned} h_1(x_3) = & 446201620912851710794055315043558727793812439040x_3^{101} \\ & - 5970091094929099291107016178576455666569981198336x_3^{100} \\ & + 54956862112783810928181826738240314636476199469056x_3^{99} \\ & - 390811972141409570262352446028389195713538504523776x_3^{98} \\ & + 2345416005757619944748010845785735225833990004408320x_3^{97} \\ & - 12296269949783674561034099631570563346793487654977536x_3^{96} \\ & + 57724805446986778564540591366311097869260812896485376x_3^{95} \\ & - 246317806151607009947614038893190006132943659073929216x_3^{94} \\ & + 965833450260597608670637228985038216735033541178353664x_3^{93} \\ & - 3507391144656252588241576000834153338016240516439009280x_3^{92} \\ & + 11864289574806461749157687866946545721640388480080753216x_3^{91} \end{aligned}$$

$$\begin{aligned}
& -37549398669469064618491404250349251298103313074223249536x_3^{90} \\
& +111556403718909975585042193985610935450985330327798025296x_3^{89} \\
& -311910444204478491525686781524742216649919870695292936208x_3^{88} \\
& +822280356201276757927422346722674238555203030542292664576x_3^{87} \\
& -2046807634124746479321033866947971592607665709250896466720x_3^{86} \\
& +4815217751927711346791066364373414448388380533686308362728x_3^{85} \\
& -10713154223781970681173785420072062491556536536976223630728x_3^{84} \\
& +22549161222430386183608267502748067727666572080394217257988x_3^{83} \\
& -44907345465673484752677835549915381616308978558104109929360x_3^{82} \\
& +84618521132058565642549103143785317869429133164796407117101x_3^{81} \\
& -150844185455615567255483714719341206736546450798252724485465x_3^{80} \\
& +254364597537871570975284759118655801254332547323430302651150x_3^{79} \\
& -405743116891842974585208861357409219861363395354878349480024x_3^{78} \\
& +612380460693768440871773749073945732272186880021261998877349x_3^{77} \\
& -875113963128364673685794098870412312567351427548775200004993x_3^{76} \\
& +1185630843249395191547875465219341425290948445000653446938823x_3^{75} \\
& -1526156016359413728700771221981215402722504361600304503707071x_3^{74} \\
& +1872140311833306055253112547604404320557478273540474816242205x_3^{73} \\
& -2197142616370071997752676286281654178936005418170556941864365x_3^{72} \\
& +2477660626761403597845544417084070514622384049066587378036896x_3^{71} \\
& -2695279129437608065469390397221342766341704813704469316565954x_3^{70} \\
& +2835433701981662882068588945651386104804048704393507162564893x_3^{69} \\
& -2884252164405306448117386421259048268986488851508923254464557x_3^{68} \\
& +2827426102309617415089093304345547856770953623242517510626736x_3^{67} \\
& -2653031260591475199015786134718569035816857413556039498720084x_3^{66} \\
& +2358669599996507371239530783357053968184783637965137255750024x_3^{65} \\
& -1957846841895078664510239606394345202411065376438607044585244x_3^{64} \\
& +1482575111628520660082626404685754443506000630228282037525432x_3^{63} \\
& -977867340837642392768018881489052493366220469043864074527732x_3^{62} \\
& +491766472172297444603170160453087191972251139268914030745220x_3^{61} \\
& -64289892666775037219425989232698561842251433476705352663704x_3^{60} \\
& -278945493495224974314286977888017233682365342439482314928900x_3^{59} \\
& +526430447145328824729273526640420290370517101891583559600064x_3^{58} \\
& -678379950713096427271349581801112479268074670188010049765152x_3^{57} \\
& +740804443402906260135865009076566078898436726677260064209660x_3^{56} \\
& -723437773277172187573344151721247865703874576057058786645840x_3^{55} \\
& +641035988242594321731767126475011901128999801733089421531572x_3^{54} \\
& -512246964568992242392698713221172537445964128081045582227348x_3^{53} \\
& +359435224624336029210539130176048914537249135075824824148656x_3^{52} \\
& -207440334442393198667039888458264495684556945767681966433896x_3^{51} \\
& +75710441609060446524728615339027145373100155489714689483180x_3^{50} \\
& +21604102072091727758542772876876306504838293754672061494810x_3^{49} \\
& -80248321816710686802873660978717285967176547645289112968438x_3^{48} \\
& +103727618132515579179402773133416119620715279258622028072932x_3^{47} \\
& -100868755076088189226682412281496766781111480676495550161140x_3^{46} \\
& +82258536089640272288149509294477219539005077695487522749302x_3^{45} \\
& -57970183302251603419654961306043425453455813053820477148466x_3^{44} \\
& +34566800668606464343700458091411364463430345415525433898210x_3^{43} \\
& -16383644568171057304816744132324477326999822644552147091878x_3^{42} \\
& +4136127404028507683677730195509342689102464969628360294162x_3^{41} \\
& +2448626876880142407515132852862076928913043337935695489050x_3^{40} \\
& -5091766004144148182025039566270136471217289325488871960720x_3^{39} \\
& +5211357311516214470643681545481350157748897861517575943216x_3^{38} \\
& -4205038938600327672804921451310081054705566281185567057186x_3^{37} \\
& +2880449505767857832367939657821857907905008739339817966542x_3^{36} \\
& -1722083524791071937275223955515869701707457252145375576992x_3^{35} \\
& +883624776677143096193640273337726641926852894182302897924x_3^{34} \\
& -365844093248197813880318931085013840771534850915964839168x_3^{33} \\
& +91100779597310425911980208315659416104663130431912801252x_3^{32} \\
& +27847050547862877200320462747908632404588568320607377368x_3^{31} \\
& -62044289670560951692846051883485532537357245491675755676x_3^{30} \\
& +58353887299451282295931377670775776235048712427538410548x_3^{29}
\end{aligned}$$

$$\begin{aligned}
& -42801139152078013889617819216603291344568555788701334096x_3^{28} \\
& +27298429710753168330147448633665498685113651255938321004x_3^{27} \\
& -15754306291891017136056764020082795793908703387391423800x_3^{26} \\
& +8385092507307451984906201504724689649995181449911987384x_3^{25} \\
& -4158959074188012109187108793403217630081388694570956516x_3^{24} \\
& +1933988921439293195469245555839057789403000884600297488x_3^{23} \\
& -846254668982236036130963702540888323950320830947175956x_3^{22} \\
& +349173361624441055757559121760687306723274054819456068x_3^{21} \\
& -136002653968064858424211455167796115036642943021590992x_3^{20} \\
& +50025097256033736148221913546357943308441010505896292x_3^{19} \\
& -17373295515964501643797136024825465698321434682820076x_3^{18} \\
& +5693288809915569850120359913933561680515494371702321x_3^{17} \\
& -1758699778656065182425515796649314364989143766487161x_3^{16} \\
& +511400484439012231385964862157738050955388834378366x_3^{15} \\
& -139733354682245393321428617402456758035160677933092x_3^{14} \\
& +35796740615382398885298967204849897416534790629933x_3^{13} \\
& -8574371869161530658987820760170185731254834290789x_3^{12} \\
& +1913914136459751675483098693532003485746543254063x_3^{11} \\
& -396457437890074706831914824872045975693198088483x_3^{10} \\
& +75819734083435665076464730202188368026690437353x_3^9 \\
& -13299684783778380566640907764584095732685621109x_3^8 \\
& +2121856171516406265190177954088620136793636144x_3^7 \\
& -304497055879317382444174963304459359118141166x_3^6 \\
& +38715888461462869028868892744214611570993773x_3^5 \\
& -4268028432749925091690300460592874748842233x_3^4 \\
& +394868683968983093689505159570909541213552x_3^3 \\
& -29044225364799470649873570312140343108960x_3^2 \\
& +1526222320849667867688684374054814420480x_3 \\
& -43316614062737407568779520450405292288.
\end{aligned}$$

Solving $h_1(x_3) = 0$ numerically, we find five positive and two negative solutions, which are given approximately by $x_3 \approx 0.18639713$, $x_3 \approx 0.34082479$, $x_3 \approx 1.1408077$, $x_3 \approx 1.4812096$, $x_3 \approx 2.3587740$ and $x_3 \approx -0.80052360$, $x_3 \approx -0.75956931$. Moreover, real solutions of the system $\{g_0 = 0, g_1 = 0, g_2 = 0, u_2 = u_3, g_4 = 0, g_5 = 0\}$ with $u_0 u_1 u_2 u_3 x_2 x_3 \neq 0$ have the form

$$\begin{aligned}
& \{u_0 \approx 0.19486610, u_1 \approx 0.15924707, u_2 = u_3 \approx 0.17659278, x_2 \approx 1.0016957, x_3 \approx 0.18639713\}, \\
& \{u_0 \approx 0.37393672, u_1 \approx 0.21919659, u_2 = u_3 \approx 1.1626641, x_2 \approx 1.0087519, x_3 \approx 0.34082479\}, \\
& \{u_0 \approx 0.84212893, u_1 \approx 0.10872180, u_2 = u_3 \approx 0.178610960, x_2 \approx 0.58058297, x_3 \approx 1.1408077\}, \\
& \{u_0 \approx 1.5046268, u_1 \approx 1.4017692, u_2 = u_3 \approx 0.23783921, x_2 \approx 1.0550442, x_3 \approx 1.4812096\}, \\
& \{u_0 \approx 2.3653572, u_1 \approx 0.16989789, u_2 = u_3 \approx 0.26106774, x_2 \approx 1.6915562, x_3 \approx 2.3587740\}
\end{aligned}$$

and

$$\begin{aligned}
& \{u_0 \approx 1.3387553, u_1 \approx 0.24195763, u_2 = u_3 \approx 0.34572988, x_2 \approx 3.8653526, x_3 \approx -0.80052360\}, \\
& \{u_0 \approx 1.1975502, u_1 \approx 0.69035070, u_2 = u_3 \approx 0.37629335, x_2 \approx 3.8406206, x_3 \approx -0.75956931\}.
\end{aligned}$$

Therefore, we obtain five Einstein metrics which are non-naturally reductive by Proposition 5.5. We also see that these five metrics are non-isometric each other, by computing the scale invariants (cf. [AC]).

Consider now the case $(x_3 - 1)(17x_3 - 3)(105x_3^2 - 180x_3 + 43) = 0$. For $105x_3^2 - 180x_3 + 43 = 0$, we get that $x_2 = 1$, $u_0 = u_1 = x_3$ and $35x_3 + 43u_2 - 60 = 0$, as solutions of the system of equations $\{g_0 = 0, g_1 = 0, g_2 = 0, u_2 = u_3, g_4 = 0, g_5 = 0\}$ with $u_0 u_1 u_2 u_3 x_2 x_3 \neq 0$. It follows that the following values are real solutions of the homogeneous Einstein equation:

$$\begin{aligned}
& \{u_0 = u_1 = x_3 = \frac{1}{105} (90 - \sqrt{3585}), u_2 = u_3 = \frac{1}{129} (90 + \sqrt{3585}), x_2 = 1\}, \\
& \{u_0 = u_1 = x_3 = \frac{1}{105} (90 + \sqrt{3585}), u_2 = u_3 = \frac{1}{129} (90 - \sqrt{3585}), x_2 = 1\}.
\end{aligned}$$

For $x_3 = 1$, we see that $(x_2 - 1)(319x_2^3 - 585x_2^2 + 298x_2 - 46) = 0$, $u_0 = u_2 = x_2$ and $23u_1 - 638x_2^3 + 1808x_2^2 - 1513x_2 + 320 = 0$. In this case we obtain the following solutions:

$$\begin{aligned} \{u_0 = u_2 = u_3 = x_2 \approx 0.32447061, u_1 \approx 0.10305040, x_3 = 1\}, \\ \{u_0 = u_2 = u_3 = x_2 \approx 0.40093736, u_1 \approx 1.6130682, x_3 = 1\}, \\ \{u_0 = u_2 = u_3 = x_2 \approx 1.1084478, u_1 \approx 0.19842410, x_3 = 1\} \end{aligned}$$

and $u_0 = u_1 = u_2 = u_3 = x_2 = x_3 = 1$. Similarly, for $17x_3 - 3 = 0$, we see that $x_2 = 1$ and $u_0 = u_1 = u_2 = u_3 = x_3 = 3/17$. According to Proposition 6.4, these values induce only naturally reductive Einstein metrics.

Case of $2u_2u_3x_2^2 + u_2u_3 - x_2^2 = 0$.

We consider the polynomial ring $R = \mathbb{Q}[z, u_0, u_1, u_2, u_3, x_2, x_3]$ and an ideal I generated by polynomials $\{g_0, g_1, g_2, 2u_2u_3x_2^2 + u_2u_3 - x_2^2, g_4, g_5, z u_0 u_1 u_2 u_3 x_2 x_3 - 1\}$. We take a lexicographic ordering $>$, with $z > u_0 > u_1 > u_2 > u_3 > x_2 > x_3$ for a monomial ordering on R . Then, a Gröbner basis for the ideal I contains a polynomial of x_3 , given by $(129x_3^2 - 180x_3 + 35)h_2(x_3)$, where $h_2(x_3)$ is a polynomial of degree 62 of the following form:

$$\begin{aligned} h_2(x_3) = & 7176902311937267597352960000x_3^{62} - 63935286731053017507053568000x_3^{61} \\ & + 399843715184474955262341120000x_3^{60} - 1965764640832898839737014200320x_3^{59} \\ & + 8193254275314202221366521370624x_3^{58} - 29792334645086350198275186421248x_3^{57} \\ & + 96312668127950023620573089912976x_3^{56} - 279881292674846037180105828690048x_3^{55} \\ & + 736814302624650073499123557784400x_3^{54} - 1765609824944028233625757684850592x_3^{53} \\ & + 3858859512197506584743793325704936x_3^{52} - 7693673293317379815378916857414144x_3^{51} \\ & + 13978077490223112750713848520000136x_3^{50} - 23093289922724516044177470806375016x_3^{49} \\ & + 34590603147607745081071881647260329x_3^{48} - 46777732301916799546957263457901784x_3^{47} \\ & + 56736923977077571150912980537328233x_3^{46} - 61045239892045312401412474185577596x_3^{45} \\ & + 57108713980664370406222711634860701x_3^{44} - 44533258600030824329782702723403172x_3^{43} \\ & + 25754099801447208802135893114691377x_3^{42} - 5506773889529575627395134456923560x_3^{41} \\ & - 10900969369323449326468798316973849x_3^{40} + 19761681445815159942981174871733088x_3^{39} \\ & - 20188941574110618150578705304430161x_3^{38} + 14202358308412638850212352987090788x_3^{37} \\ & - 5742232611467996731383311668741917x_3^{36} - 1570190865515015457366830818567044x_3^{35} \\ & + 5648740711606386591873392674677879x_3^{34} - 5948880658615999334374796571219144x_3^{33} \\ & + 3868106929529148762149605373828154x_3^{32} - 1258995580792601908577943150171216x_3^{31} \\ & - 853879665000749089689937169600598x_3^{30} + 1715750052119667014938302077428104x_3^{29} \\ & - 1499771919293409567273925312020942x_3^{28} + 912888495260506476555714495569400x_3^{27} \\ & - 254675582893363617963644083458150x_3^{26} - 156524700943028756585375788114864x_3^{25} \\ & + 250591779632840858079012274593070x_3^{24} - 210754892342268589669423384800992x_3^{23} \\ & + 104847524519386925269074792690078x_3^{22} - 16046875591916438782483046815032x_3^{21} \\ & - 19822089010062331168255611933090x_3^{20} + 27934823512978080917038217731864x_3^{19} \\ & - 18891540266011228264345732880890x_3^{18} + 7806076572680194957966216721384x_3^{17} \\ & - 1269024611811072417466760739747x_3^{16} - 1623952795000087557877271247768x_3^{15} \\ & + 1876612307555759003085287593757x_3^{14} - 1325495568850663913046822215564x_3^{13} \\ & + 722686827400045498790015516321x_3^{12} - 329147016934629053928253191636x_3^{11} \\ & + 129532305407715005005113350181x_3^{10} - 44529061623094804552954953960x_3^9 \\ & + 13469000037841152898081791595x_3^8 - 3593330620975215538779860800x_3^7 \\ & + 840650703590361114731290675x_3^6 - 171221634223124842794559500x_3^5 \\ & + 29979979441425878491638375x_3^4 - 4380052206593637030292500x_3^3 \\ & + 514712484479377144996875x_3^2 - 45016029840810490875000x_3 + 2277208974357528750000. \end{aligned}$$

Solving $h_2(x_3) = 0$ numerically, we find four positive and two negative solutions, which are given approximately by $x_3 \approx 1.0032864$, $x_3 \approx 0.25857941$, $x_3 \approx 1.3899161$, $x_3 \approx 1.4812096$, $x_3 \approx 1.0082270$ and $x_3 \approx -0.71677982$, $x_3 \approx -0.70718824$. In particular, real solutions of the system $\{g_0 = 0, g_1 = 0, g_2 = 0, 2u_2u_3x_2^2 + u_2u_3 - x_2^2 = 0, g_4 = 0, g_5 = 0\}$ with $u_0 u_1 u_2 u_3 x_2 x_3 \neq 0$ have the form

$$\begin{aligned} \{u_0 \approx 0.35854650, u_1 \approx 0.10372795, u_2 \approx 0.42074677, u_3 \approx 0.22748595, x_2 \approx 0.34405535, x_3 \approx 1.0032864\}, \\ \{u_0 \approx 0.35854650, u_1 \approx 0.10372795, u_2 \approx 0.22748595, u_3 \approx 0.42074677, x_2 \approx 0.34405535, x_3 \approx 1.0032864\}, \\ \{u_0 \approx 0.27724447, u_1 \approx 0.19525496, u_2 \approx 1.4284934, u_3 \approx 0.23400191, x_2 \approx 1.0042307, x_3 \approx 0.25857941\}, \\ \{u_0 \approx 0.27724447, u_1 \approx 0.19525496, u_2 \approx 0.23400191, u_3 \approx 1.4284934, x_2 \approx 1.0042307, x_3 \approx 0.25857941\}, \\ \{u_0 \approx 1.5127009, u_1 \approx 0.18053147, u_2 \approx 1.3241010, u_3 \approx 0.28487472, x_2 \approx 1.2393058, x_3 \approx 1.3899161\}, \\ \{u_0 \approx 1.5127009, u_1 \approx 0.18053147, u_2 \approx 0.28487472, u_3 \approx 1.3241010, x_2 \approx 1.2393058, x_3 \approx 1.3899161\}, \\ \{u_0 \approx 0.47336119, u_1 \approx 1.5881009, u_2 \approx 0.25625661, u_3 \approx 0.55478019, x_2 \approx 0.44569962, x_3 \approx 1.0082270\}, \\ \{u_0 \approx 0.47336119, u_1 \approx 1.5881009, u_2 \approx 0.55478019, u_3 \approx 0.25625661, x_2 \approx 0.44569962, x_3 \approx 1.0082270\} \end{aligned}$$

and

$$\begin{aligned} &\{u_0 \approx 1.0325160, u_1 \approx 0.34059800, u_2 \approx 1.1055012, u_3 \approx 0.435626061, x_2 \approx 3.6160749, x_3 \approx -0.71677982\}, \\ &\{u_0 \approx 1.0325160, u_1 \approx 0.34059800, u_2 \approx 0.435626061, u_3 \approx 1.1055012, x_2 \approx 3.6160749, x_3 \approx -0.71677982\}, \\ &\{u_0 \approx 1.0018570, u_1 \approx 0.49052811, u_2 \approx 1.0713232, u_3 \approx 0.44960349, x_2 \approx 3.6248195, x_3 \approx -0.70718824\}, \\ &\{u_0 \approx 1.0018570, u_1 \approx 0.49052811, u_2 \approx 0.44960349, u_3 \approx 1.0713232, x_2 \approx 3.6248195, x_3 \approx -0.70718824\}. \end{aligned}$$

Based now on Proposition 6.4, we conclude that these eight Einstein metrics are non-naturally reductive. Notice that metrics obtained by exchanging u_2 and u_3 are isometric. A final computation of the induced scale invariants allows us to deduce that there are four Einstein metrics which are non-isometric each other.

For $129x_3^2 - 180x_3 + 35 = 0$, we get that $x_2 = 1$, $u_0 = u_1 = x_3$, $24u_3x_3 + 105u_3^2 - 180u_3 + 35 = 0$ and $8x_3 + 35u_2 + 35u_3 - 60 = 0$ for solutions of the system of equations $\{g_0 = 0, g_1 = 0, g_2 = 0, 2u_2u_3x_2^2 + u_2u_3 - x_2^2 = 0, g_4 = 0, g_5 = 0\}$ with $u_0 u_1 u_2 u_3 x_2 x_3 \neq 0$. Thus, in this case solutions of the homogeneous Einstein equation are given by

$$\begin{aligned} &\{u_0 = u_1 = x_3 = \frac{1}{129} (90 - \sqrt{3585}) = u_3, u_2 = \frac{1}{105} (90 + \sqrt{3585}), x_2 = 1\}, \\ &\{u_0 = u_1 = x_3 = \frac{1}{129} (90 - \sqrt{3585}) = u_2, u_3 = \frac{1}{105} (90 + \sqrt{3585}), x_2 = 1\}, \\ &\{u_0 = u_1 = x_3 = \frac{1}{129} (90 + \sqrt{3585}) = u_3, u_2 = \frac{1}{105} (90 - \sqrt{3585}), x_2 = 1\}, \\ &\{u_0 = u_1 = x_3 = \frac{1}{129} (90 + \sqrt{3585}) = u_2, u_3 = \frac{1}{105} (90 - \sqrt{3585}), x_2 = 1\}. \end{aligned}$$

By Proposition 6.4, these solutions give rise to naturally reductive Einstein metrics.

REFERENCES

- [AP] D. V. Alekseevsky, A. M. Perelomov, *Invariant Kähler-Einstein metrics on compact homogeneous spaces*, Funct. Anal. Appl. 20 (3), (1986), 171–182.
- [AnC] S. Anastassiou, I. Chrysikos, *The Ricci flow approach to homogeneous Einstein metrics on flag manifolds*, J. Geom. Phys. 61 (2011), 1587–1600.
- [AC] A. Arvanitoyeorgos, I. Chrysikos, *Invariant Einstein metrics on generalized flag manifolds with four isotropy summands*, Ann. Glob. Anal. Geom. 37 (2), (2010), 185–219.
- [AMS] A. Arvanitoyeorgos, K. Mori, Y. Sakane, *Einstein metrics on compact Lie groups which are not naturally reductive*, Geom. Dedicata, 160, (2012), 261–285.
- [B] A. L. Besse, *Einstein Manifolds*, Springer-Verlag, Berlin, 1986.
- [CL] Z. Chen, K. Liang, *Non-naturally reductive Einstein metrics on the compact simple Lie group F_4* , Ann. Glob. Anal. Geom. 46, (2014), 1103–1115.
- [C1] I. Chrysikos, *Homogeneous Einstein Metrics on Generalized Flag Manifolds*, Ph.D Thesis (Greek), University of Patras, 2011 (electronic version: <http://nemertes.lis.upatras.gr/jspui/handle/10889/4418?locale=en>).
- [C2] I. Chrysikos, *Flag manifolds, symmetric t-triples and Einstein metrics*, Diff. Geom. Appl. Vol. 30, (6), (2012), 642–659.
- [CS] I. Chrysikos, Y. Sakane, *The classification of homogeneous Einstein metrics on flag manifolds with $b_2(M) = 1$* , Bull. Sci. Math. 138, (2014), 665–692.
- [DZ] J. E. D’Atri, W. Ziller, *Naturally reductive metrics and Einstein metrics on compact Lie groups*, Mem. Am. Math. Soc. 18 (215), (1979), 1–72.
- [FdV] H. Freudenthal, H. de Vries, *Linear Lie Groups*, Academic Press, New York, 1969.
- [GLP] G. W. Gibbons, H. Lü, C. N. Pope, *Einstein metrics on group manifolds and cosets*, J. Geom. Phys. 61 (5), (2011), 947–960.
- [GOV] V. V. Gortzevich, A. L. Onishchik, E. B. Vinberg, *Lie Groups and Lie Algebras III*, Encyclopaedia of Mathematical Sciences, Vol. 20, Springer-Verlag, Berlin, 1993.
- [K] M. Kimura, *Homogeneous Einstein metrics on certain Kähler C-spaces*, Adv. Stud. Pure. Math. 18-I (1990), 303–320.
- [M] K. Mori, *Left Invariant Einstein Metrics on $SU(N)$ that are Not Naturally Reductive*, Master Thesis (in Japanese), Osaka University 1994, English translation Osaka University RPM 96 –10 (preprint series), 1996.
- [PS] J.-S. Park, Y. Sakane, *Invariant Einstein metrics on certain homogeneous spaces*, Tokyo J. Math. 20 (1) (1997), 51–61.
- [S] Y. Sakane, *Homogeneous Einstein metrics on flag manifolds*, Lobachevskii J. Math. 4 (1999), 71–87.
- [WZ] M. Wang, W. Ziller, *Existence and non-existence of homogeneous Einstein metrics*, Invent. Math. 84, (1986), 177–194.

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